LECTURE #8 - Phasors and Impedance

When a circuit is fed with a sinusoidal input:

- All voltages and currents will be sinusoidal and have the same frequency as the input
- The amplitude will be scaled and the phase will be shifted.

When circuit excitation is complex, amplitude scaling and phase shifting can easily be done by multiplying the scale values and adding exponents.

The transfer function is the ratio of the circuit output to the circuit input. It is generally complex

$$H = \frac{V_o}{V_{in}} \quad \text{or} \quad H = \frac{I_{R_2}}{V_{in}} \quad \text{or} \quad H = \frac{I_o}{I_{in}}.$$

To get the output, multiply the input by the transfer function.

A phasor is a complex number that represents a sinusoid

$v(t) = V_m \cos(\omega t + \phi)$ ⟵ Sinusoid

$v(t) = V_m e^{j(\omega t + \phi)}$ ⟵ we found we can streamline circuit analysis by using complex excitation

$v(t) = V_m e^{j\omega t} e^{j\phi}$

$V = V_m e^{j\phi}$ ⟵ Dump the $e^{j\omega t}$ because it will be part of every voltage and current.

$V = V_m \angle \phi$ ⟵ phasor notation

$$\underbrace{v(t) = V_m \cos(\omega t + \phi)}_{\text{Time domain representation}} \Longleftrightarrow \underbrace{V = V_m \angle \phi}_{\text{phasor domain representation}}$$

↑

EQ 9.25 Pg 379

For inductor, $V = L\dfrac{di}{dt}$

For capacitor, $i = c\dfrac{dv}{dt}$

If we can represent v by a phasor how do we represent dv/dt ?

$$v(t) = V_m \cos(\omega t + \phi)$$

$$\frac{dv}{dt} = -V_m \omega \sin(\omega t + \phi) = -V_m \omega \cos(\omega t + \phi - \pi/2)$$

$$\frac{dv}{dt} = -V_m \omega e^{j(\omega t + \phi - \pi/2)}$$ ← represent dv/dt as counter clockwise vector

$$\frac{dv}{dt} = -V_m \omega e^{j\omega t} e^{j\phi} \underbrace{e^{-j\pi/2}}$$

$$\hookrightarrow \cancelto{0}{\cos(-\pi/2)} + j\,\cancelto{-1}{\sin(-\pi/2)} = -j$$

so...

$$\frac{dv}{dt} = V_m j\omega e^{j\omega t} e^{j\phi}$$

remove $e^{j\omega t}$ for phasors

$$= V_m \cdot j\omega e^{j\phi} = j\omega V_m e^{j\phi} = j\omega V$$

So we represent derivative of sinwave by multiplying the original phasor by $j\omega$

To get integral of V, divide by $j\omega$

Let's get back to inductors and capacitors

inductor, $v = L \frac{di}{dt}$

or, using phasors

$$V = j\omega L I$$

or the impedance of an inductor is:

$$Z_L = \frac{V_L}{I_L} = j\omega L$$

Capacitor

$$i = C \frac{dv}{dt}$$

so $I = C j\omega V$

or $Z_c = \frac{V_c}{I_c} = \frac{1}{j\omega C}$

Three ways to represent a phasor

$$Z = x + jy$$
$$Z = r \angle \phi$$
$$Z = re^{j\phi}$$

Example, PR 9.27, Pg 405

determine $v(t)$ when

$$\frac{dv}{dt} + 50v + 100\int v\,dt = 110 \cdot \cos(377t - 10^\circ)$$

First: Note that $\omega = 377$ rad/sec ... 60 Hz

Convert the equation to phasor domain:

$$j\omega V + 50V + 100\frac{V}{j\omega} = 110\angle{-10^\circ}$$

$$(j\omega)^2 V + 50 j\omega V + 100V = j\omega \times 110\angle{-10^\circ}$$

$$V\left((j\omega)^2 + 50j\omega + 100\right) = j\omega \times 110\angle{-10^\circ}$$

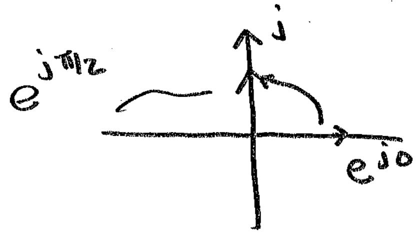
$$V = \frac{j\omega \times 110\angle{-10^\circ}}{-\omega^2 + 50j\omega + 100}$$

How do we handle the j? Solve for ϕ below

$$e^{j\phi} = \cos\phi + j\sin\phi = j$$

by inspection, $\phi = \pi/2$ or $j = e^{j\pi/2}$

$= 1\angle{90^\circ}$



or, I could convert $110\angle{-10^\circ}$ to rectangular form

$$110\angle{-10^\circ} \longrightarrow 110e^{-j\frac{10}{360}\times 2\pi} = 110\,e^{-j0.1745}$$

$$= 110\times\left[\cos(-0.1745) + j\sin(0.1745)\right]$$

and $$V = \frac{j\omega\cdot 110\left[0.9848 - j0.1736\right]}{-\omega^2 + 50j\omega + 100}$$

substitute $\omega = 377$ r/s

$$V = -0.0123 - j0.2892$$

$$= 0.2894\angle{-92.44^\circ}$$

so $V(t) = 0.289\cos(377t - 92.4^\circ)\,V$

LECTURE 8 -

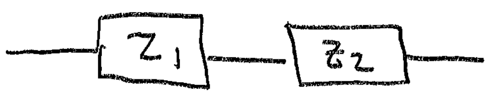
$$\text{Impedance} = \left.\frac{V}{I}\right\} \text{both phasors}$$

$\longrightarrow Z_L = j\omega L$

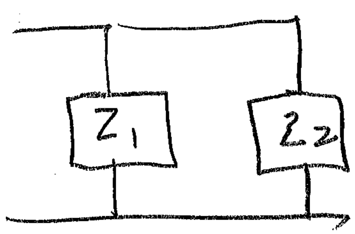
$\longrightarrow Z_C = \dfrac{1}{j\omega C}$

sinusoidal signals only

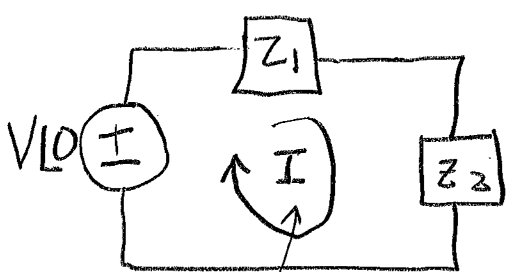
$\longrightarrow Z_R = R$



$$Z_s = Z_1 + Z_2$$



$$Z_p = \frac{Z_1 Z_2}{Z_1 + Z_2}$$



$$-V + I \cdot Z_1 + I Z_2 = 0$$

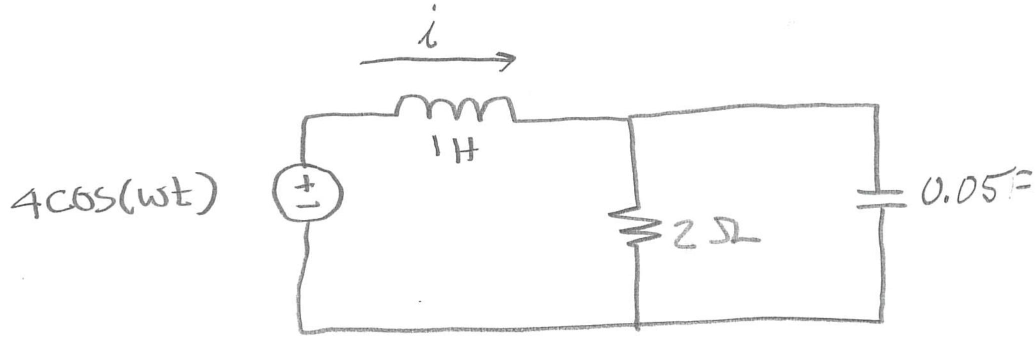
V and I are phasors

KVL holds
KCL holds
Superposition holds
⋮

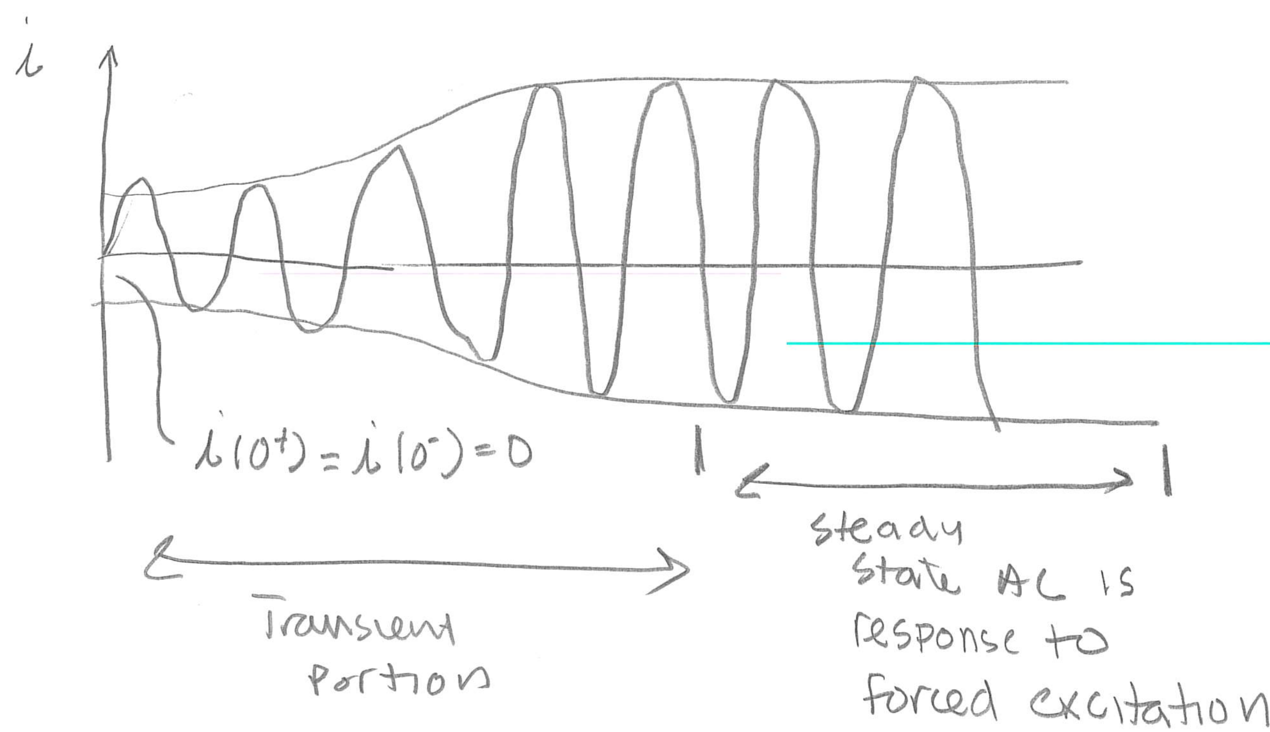
No Transient → Steady-state AC

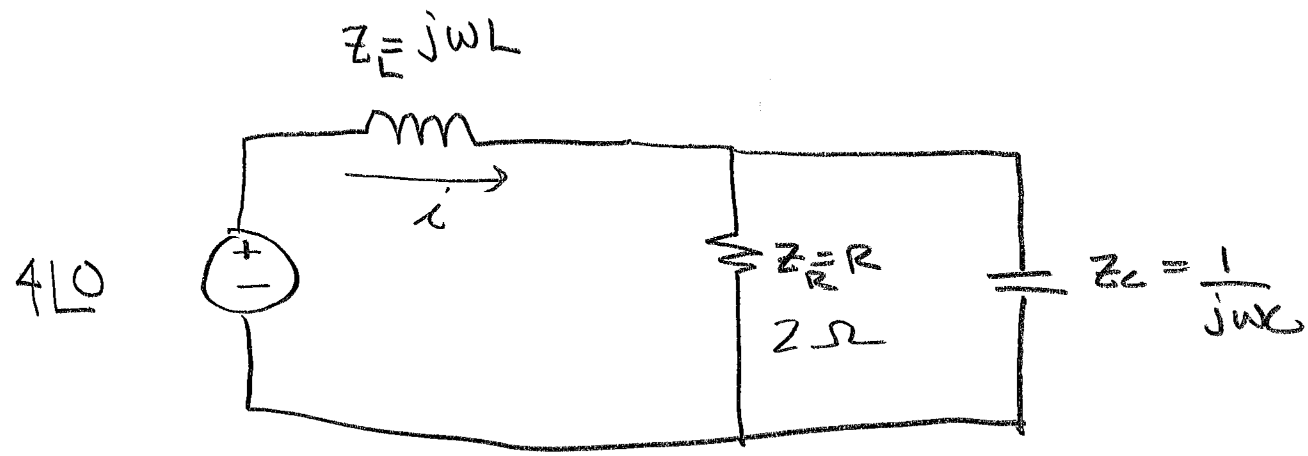
PROBLEM 9.40

find $i(t)$ at $\omega = 0, 1, 5, 10, \infty$ r/s



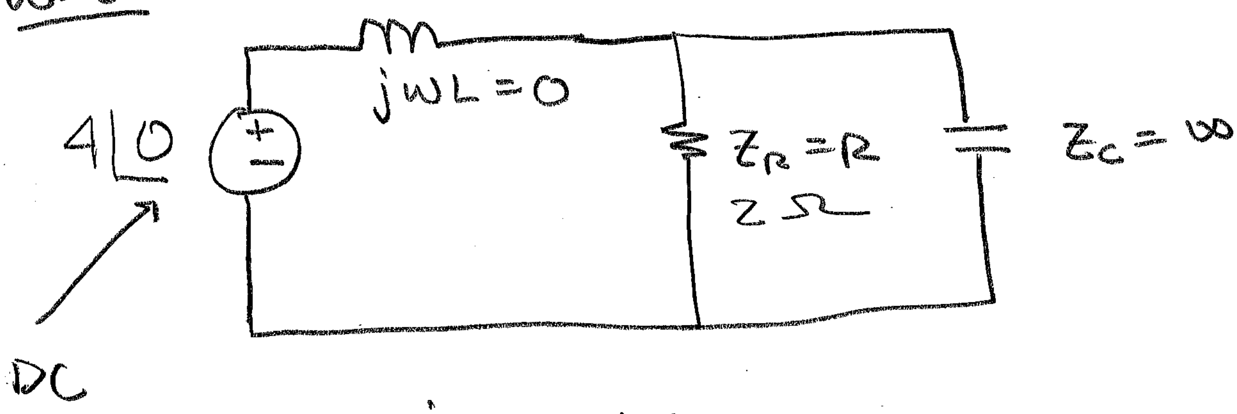
Say we started with everything turned off





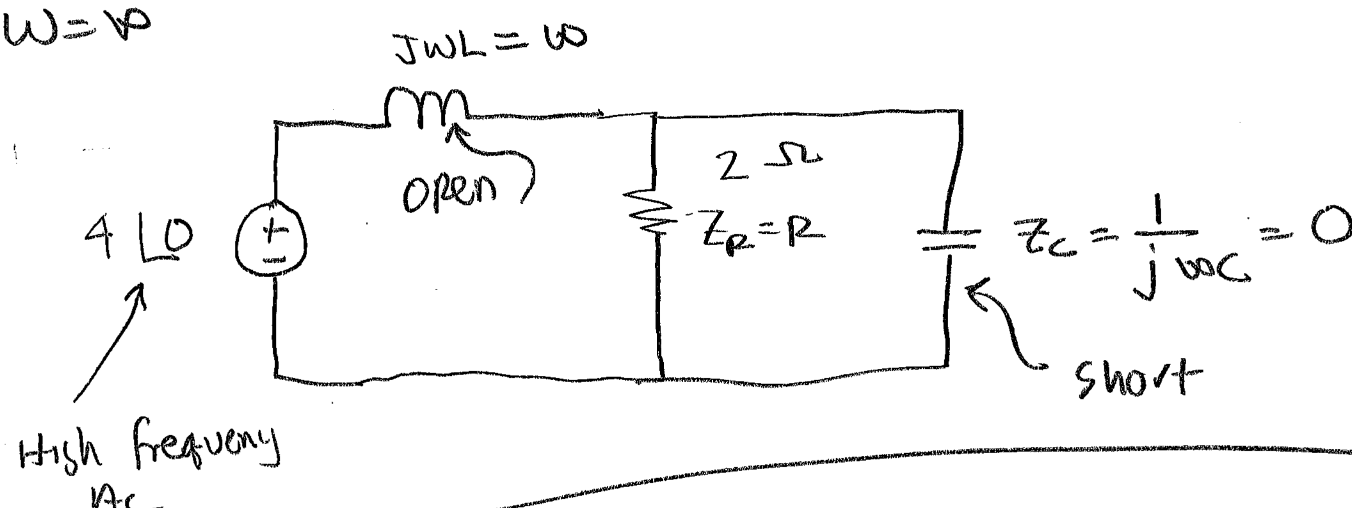
Get current at $\omega = 0$ and $\omega = \infty$ by inspection

<u>$\omega = 0$</u>



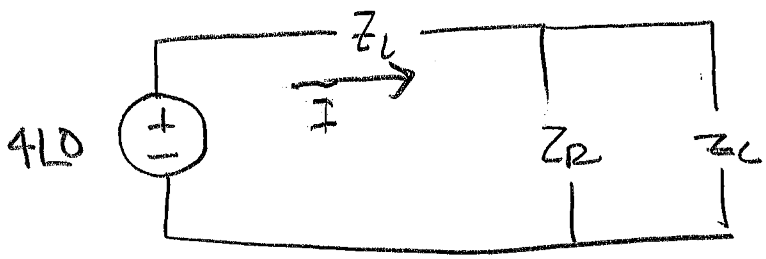
So $i = \frac{4V}{2\Omega} = 2A$ and it's DC

$\omega = \infty$

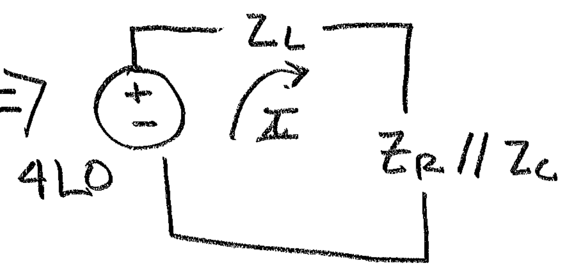


So you see response changes with frequency useful ??

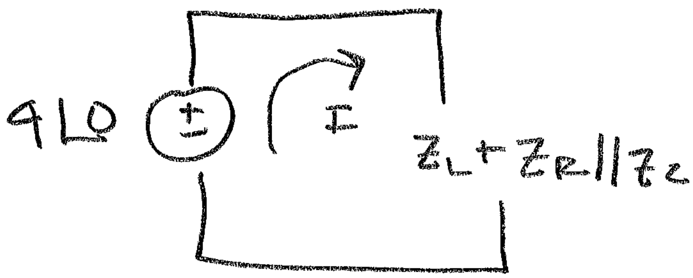
Now solve for $i(t)$ at $\omega = 1, 5, 10$ r/s



$\Rightarrow$



$\Rightarrow$



$$I = \frac{4}{Z_L + Z_R // Z_C} = \frac{4}{j\omega L + \dfrac{R \times \frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}}$$

$$= \frac{4\left(R + \frac{1}{j\omega C}\right)}{j\omega L\left(R + \frac{1}{j\omega C}\right) + \dfrac{R}{j\omega C}} = \frac{4R + \dfrac{4}{j\omega C}}{j\omega LR + \dfrac{j\omega L}{j\omega C} + \dfrac{R}{j\omega C}}$$

$$= \frac{4R \cdot j\omega C + 4}{-\omega^2 LRC + j\omega L + R}$$

j^2

$$I = \frac{4(1 + j\omega RC)}{R - \omega^2 LRC + j\omega L}$$

We have an expression for I

Let's check it by substituting $\omega = 0$, $\omega = \infty$

<u>at $\omega = 0$</u>

$$I = \frac{4(1+0)}{R - 0 + 0} = \frac{4}{R} = \frac{4}{2} = 2A \checkmark$$

at $\omega = \infty$

$$I = \frac{4(1 + j\infty)}{R - \infty^2 + \infty} = \frac{4}{-\infty} = 0 \checkmark$$

Next use your calculator to compute I

ω	I	$\lvert I \rvert$	$\angle I$	$i(t)$
0	2	2	0	$2\cos(0)$
1	$1.734 - j0.703$	1.87	-22°	$1.87\cos(t - 22^\circ)$
5	$0.3168 - j0.832$	0.89	-69.14°	$0.89\cos(5t - 69.14^\circ)$
10	$0.0488 - j0.439$	0.442	-83.6°	$0.442\cos(10t - 83.6^\circ)$
∞	0	0	-90°	$0\cos(\infty t - 90^\circ)$

<u>Silly book notation:</u>

$$i(t) = 0.442\cos(10t - 83.6^\circ)$$

$\omega = 10$ (radians) in degrees

11.5

```
clear all;
close all;

R = 2;        % Set resistor to 2 Ohms.
L = 1;        % Set inductor to 1 H.
C = 0.05;     % Set capacitor to 0.5 F

% w = 1;  % Set omega to 1 radian/sec
% w = 5;  % Set omega to 1 radian/sec
w = 10;  % Set omega to 1 radian/sec

% Get component impedances at this frequency.
Zr = R;
Zl = 1j*w*L;
Zc = 1/(1j*w*C);

I = 4/(Zl + (Zr*Zc)/(Zr+Zc));    % Current phasor

% Now get the magnitude and angle.
Magnitude = abs(I);
AngleDegrees = angle(I)/(2*pi)*360;
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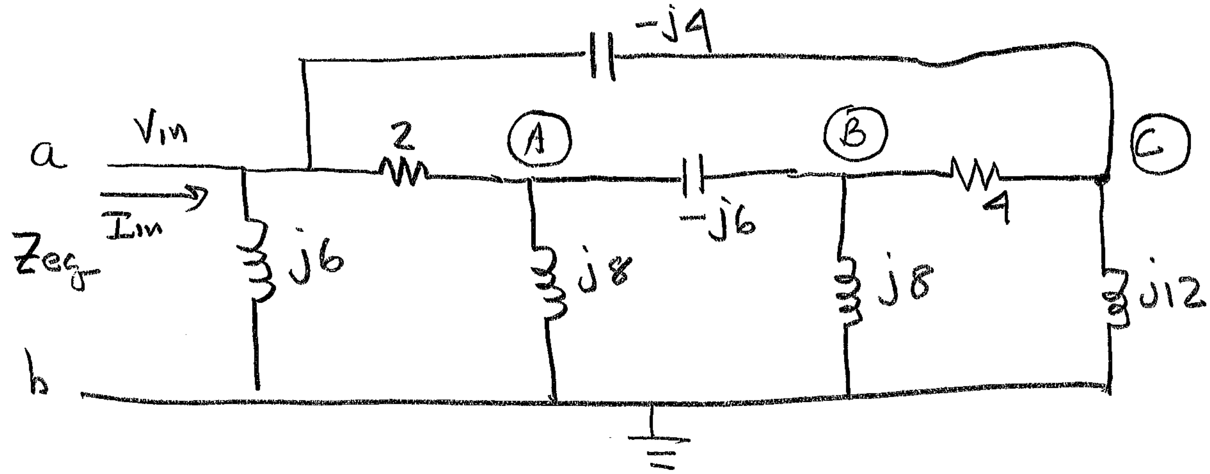
PROBLEM 9.73

Find the equivalent impedance between a and b.



Strategy - input 1 VAC
find node voltages A, B, C

$$I_{in} = \frac{V_{in}}{j6} + \frac{V_{in}-V_C}{-j4} + \frac{V_{in}-V_A}{2} \qquad \text{Then} \quad \frac{V_{in}}{I_{in}} = Z_{in}$$

KCL at input node (Σ currents leaving $= 0$)

NO need for this
Use NODES A, B, and C only

$$\frac{V_{in}}{j6} + \frac{V_{in}-V_C}{-j4} + \frac{V_{in}-V_A}{2} = I_{in}$$

or

$$-\frac{1}{2}V_A + 0\,V_B + \frac{1}{j4}V_C = -V_{in}\left[\frac{1}{j6} - \frac{1}{j4} + \frac{1}{2}\right]$$

KCL at NODE A

$$\frac{V_A - V_{in}}{2} + \frac{V_A}{j8} + \frac{V_A - V_B}{-j6} = 0$$

or

$$V_A\left[\frac{1}{2} + \frac{1}{j8} - \frac{1}{j6}\right] + \frac{V_B}{j6} + 0\,V_C = +\frac{1}{2}V_{in}$$

KCL at Node B

$$\frac{V_B - V_A}{-j6} + \frac{V_B}{j8} + \frac{V_B - V_C}{4} = 0$$

or

$$V_A \times \frac{1}{j6} + V_B\left[\frac{-1}{j6} + \frac{1}{j8} + \frac{1}{4}\right] - \frac{1}{4} V_C = 0$$

KCL at Node C

$$\frac{V_C - V_B}{4} + \frac{V_C}{j12} + \frac{V_C - V_{in}}{-j4} = 0$$

or

$$(\ )\, V_A - \frac{1}{4} V_B + V_C\left[\frac{1}{4} + \frac{1}{j12} - \frac{1}{j4}\right] = \frac{-V_{in}}{j4}$$

or

$$\begin{array}{l} \text{KCL (A)} \rightarrow \\[2em] \text{(B)} \rightarrow \\[2em] \text{(C)} \rightarrow \end{array}
\overbrace{\begin{bmatrix} \frac{1}{2} + \frac{1}{j8} - \frac{1}{j6} & \frac{1}{j6} & 0 \\ \frac{1}{j6} & -\frac{1}{j6} + \frac{1}{j8} + \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{-1}{4} & \frac{1}{4} + \frac{1}{j12} - \frac{1}{j4} \end{bmatrix}}^{\text{MATRIX A IN MATLAB}}
\overbrace{\begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix}}^{\text{MATRIX } V}
=
\overbrace{\begin{bmatrix} \frac{V_{in}}{2} \\ 0 \\ -\frac{V_{in}}{j4} \end{bmatrix}}^{\text{MATRIX } B}$$

From here I compute V_A, V_B, V_C, then follow equation for Z_{in} in our strategy section, see posted Matlab