# Complex Exponentials and Phasors

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*February 13, 2020*

*This lecture is dedicated to fred harris (capitals omitted at his insistence) who taught this material with enthusiasm and skill to multiple generations of SDSU students for over 50 years.*

**Motivation**

The purpose of this lecture is to reinforce your knowledge of complex exponentials so that you will thrive as a student and, more importantly, have the tools to continue learning throughout an exciting and interesting career. You may not fully grasp this lecture today. But it's my hope that you will keep these notes and refer to them in the future. If you can't find them, email me at bdorr@sdsu.edu and I'll send you a copy.

**Introduction**

In a circuit, a sinewave is a voltage or current – NOT a complex (or imaginary) number. There are no "imaginary" numbers rattling around in circuits. The term "imaginary number" is simply misleading! The reality is that complex numbers (which have "imaginary" parts) are commonly used for circuit analysis.

In EE 310 complex exponentials are used for AC steady state circuits. In your senior-level coursework, and when you work in industry, you will use complex exponentials to describe circuits, analog/digital filters, control systems communication signals, etc. Modern signal processing is built on a foundation of complex exponentials.

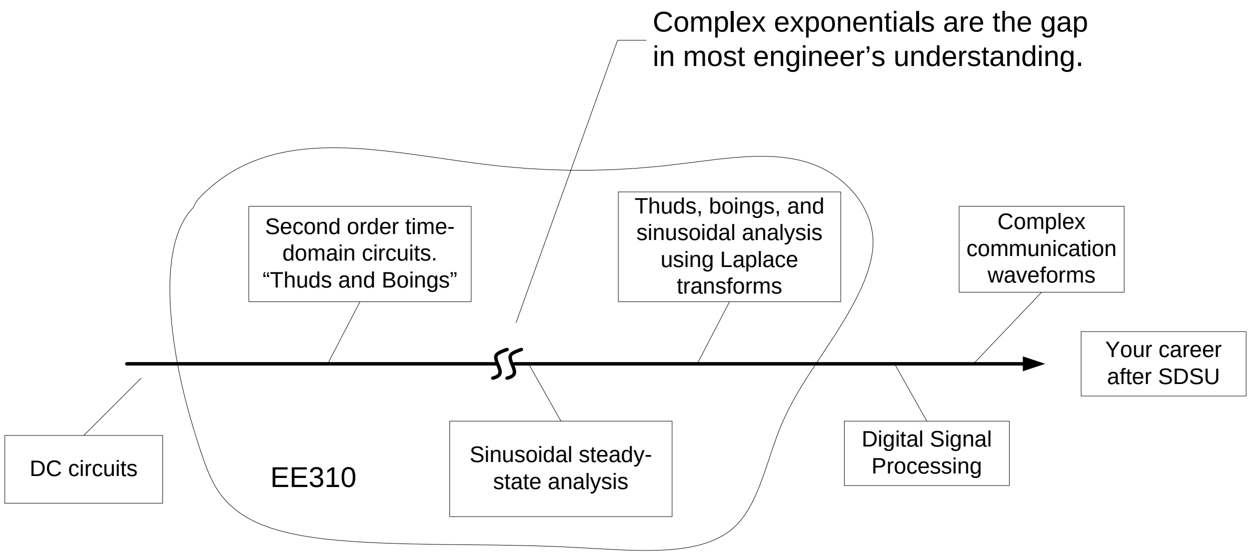


Figure 1 – Signal Processing Roadmap

## The Motivation – Circuits before 1900

Engineers, physicists, and mathematicians knew that when a sinewave was fed to a circuit, the output was also a sinewave, but it was scaled in amplitude and shifted in phase. The amplitude scaling could be handled with a simple multiplicative constant, but phase complicated things.

The differential equations describing inductors and capacitors, $v_L(t) = L \cdot \frac{di}{dt}$ and $i_C(t) = C \cdot \frac{dv}{dt}$, were well known, but differential equations had to be solved whenever a circuit was analyzed. General Electric had to electrify an entire country, and the linemen and technicians, and engineers in the field couldn't solve differential equations. A better analysis technique was needed!

## The Euler Identities

Since the Euler identities will be used throughout this lecture, they are presented here.

$$e^{j\phi} = cos(\phi) + j \cdot sin(\phi)$$

Equation 1

Equation 1 can be solved for the cosine or sine

$$cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

Equation 2

$$sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2 \cdot j}$$

Equation 3

When referring to how signals are affected by circuits at a specific frequency, $\phi$ will be a constant. For example, a signal can be shifted in phase $\phi$ radians by multiplying it by $e^{j\phi}$. When referring to time-varying signals using complex numbers, $\phi$ will be a phase angle that increases linearly with time. For example,

$$cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Equation 4 – ω is the angular frequency in radians per second. The first term in the numerator of the right-hand side has positive frequency and the second term has negative frequency.

## Graphical Representations of Complex Numbers

Consider a cosine signal using the representation of Equation 4.

$$v(t) = A \cdot cos(\omega t + \phi_1) = A \cdot \frac{e^{j(\omega t + \phi_1)} + e^{-j(\omega t + \phi_1)}}{2}$$

Equation 5

The right-hand side of Equation 5 can be represented by two vectors as shown in Figure 2. The upper vector has positive frequency and rotates counter-clockwise, and the lower vector has negative frequency and rotates clockwise. It is seen that when these vectors are added, the imaginary parts cancel, leaving a signal that moves back and forth on the real axis. Note, however, that the phase angle $\phi_1$ causes a constant phase shift as shown in the figure.

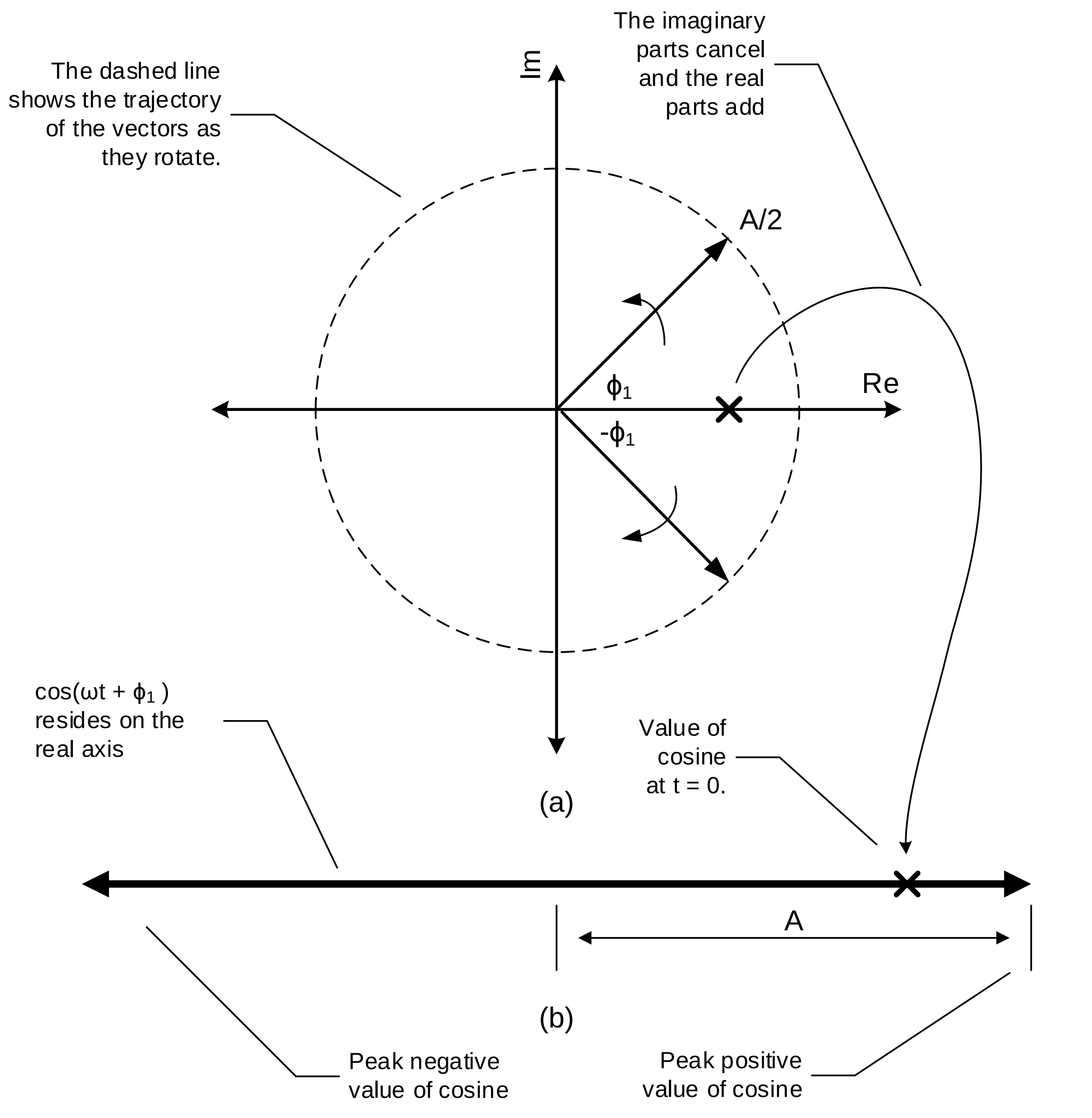


Figure 2 – (a) Cosine function represented as counter-rotating vectors from Equation 4. (b) The vector sum is a real signal with peak value A.

The phase of the cosine in Equation 6 and Figure 2 can be shifted by multiplying the upper vector by $e^{j\phi_2}$ and the lower vector by $e^{-j\phi_2}$ as shown in Figure 3. Note that when the vectors are added, the sum is still on the real axis, but it is shifted in phase.

$$v(t) = A \cdot cos(\omega t + \phi_1 + \phi_2) = A \cdot \frac{e^{j(\omega t+\phi_1+\phi_2)} + e^{-j(\omega t+\phi_1+\phi_2)}}{2}$$

Equation 6

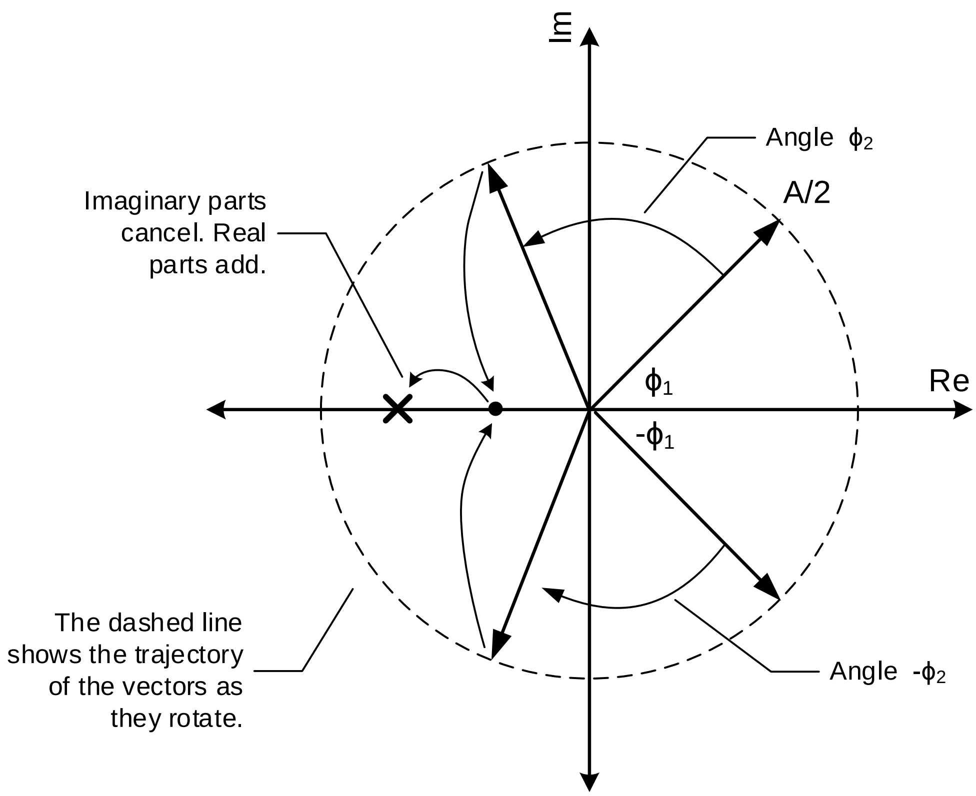


Figure 3 - Counter-rotating vectors (dashed) are phase shifted by $\phi_2$. The vector sum resides on the real axis.

Note that the upper vectors in the two figures above rotate counter-clockwise and therefore have positive frequency, and the lower vectors rotate clockwise and have negative frequency. The next section shows that when a sinewave is fed to a circuit, the circuit scales its magnitude and shifts its phase. When signals are represented as shown in Figure 2 and Figure 3, the circuit shifts the positive frequency component by a constant phase angle and the negative frequency by the negative of the same phase angle. Furthermore, we note that amplitude scaling is an even function of frequency and phase shifting is an odd function of frequency, thus insuring that the output of a circuit, when fed with a real signal, will always be a real signal.

**Analyzing Circuits Using Complex Numbers**

We wish to find the output signal resulting from feeding the circuit of Figure 4 with the signal $cos(\omega t + \phi)$. The first step is to represent the input signal as a complex exponential as shown by the vectors at the top of the figure.

$$v_{in}(t) = A \cdot cos(\omega t + \phi) = A \cdot \frac{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}}{2}$$

Equation 7

At frequency $\omega$, the circuit in Figure 4 scales the magnitude of the two vectors by constant $|H|$, and then shifts the positive frequency vector by phase angle $\gamma$ and the negative frequency vector by $-\gamma$ as shown graphically at the bottom of the figure. As in the previous examples, the imaginary parts cancel and the circuit output is a scaled and phase shifted replica of the input sinewave.

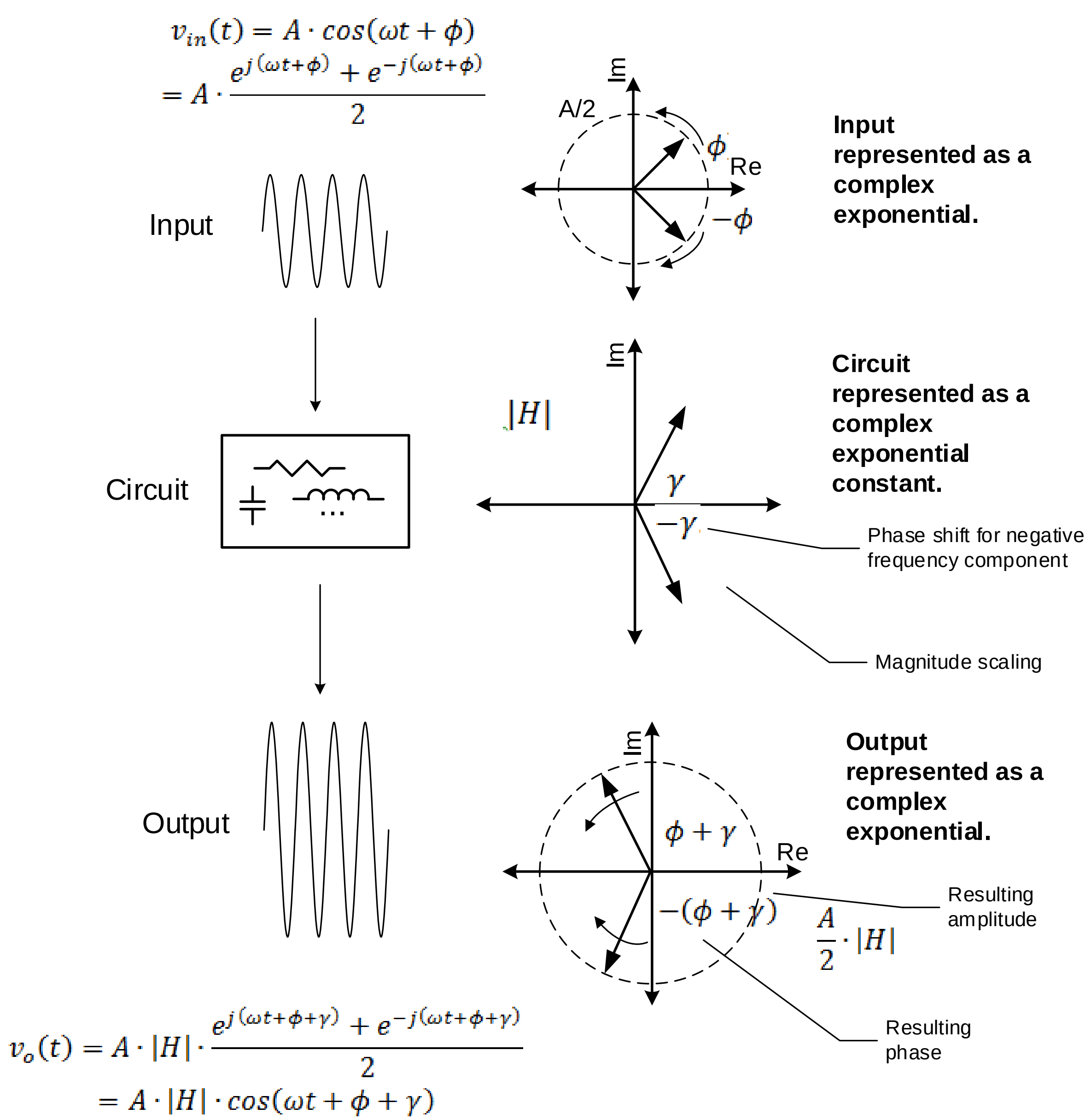


**Figure 4 – Solving a circuit problem using the Euler identities**

The resulting output signal is

$$v_o(t) = A \cdot |H| \cdot \frac{e^{j(\omega t+\phi+\gamma)} + e^{-j(\omega t+\phi+\gamma)}}{2} = A \cdot |H| \cdot cos(\omega t + \phi + \gamma)$$

**Equation 8**

Figure 5 shows how the steps associated with circuit analysis can be streamlined. First, note that for any real signal, there are always two counter-rotating, conjugate vectors. Second, note that the circuit scales the amplitude of the two vectors by the same amount. Finally, it shifts the input vectors by conjugate phase angles. The strategy is to perform calculations on the upper vector only.

As before, the input to the circuit is:

$$v_{in}(t) = A \cdot cos(\omega t + \phi) = A \cdot \frac{e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)}}{2}$$

Equation 9

Removing the lower vector gives:

$$v_{in}(t) = \frac{A}{2} \cdot e^{j(\omega t + \phi)}$$

Equation 10

The circuit scales the magnitude of the input vector by constant $|H|$, and shifts it by phase angle $\gamma$. The result is:

$$v_o(t) = \frac{A}{2} \cdot |H| \cdot e^{j(\omega t + \phi + \gamma)}$$

Equation 11

which is shown as the upper vector shown at the right of Figure 5. Finally, we restore the conjugate of the output vector, and use Euler's identity to get the output signal which is real.

$$v_o(t) = A \cdot |H| \cdot \frac{e^{j(\omega t + \phi + \gamma)} + e^{-j(\omega t + \phi + \gamma)}}{2} = A \cdot |H| \cdot cos(\omega t + \phi + \gamma)$$

Equation 12

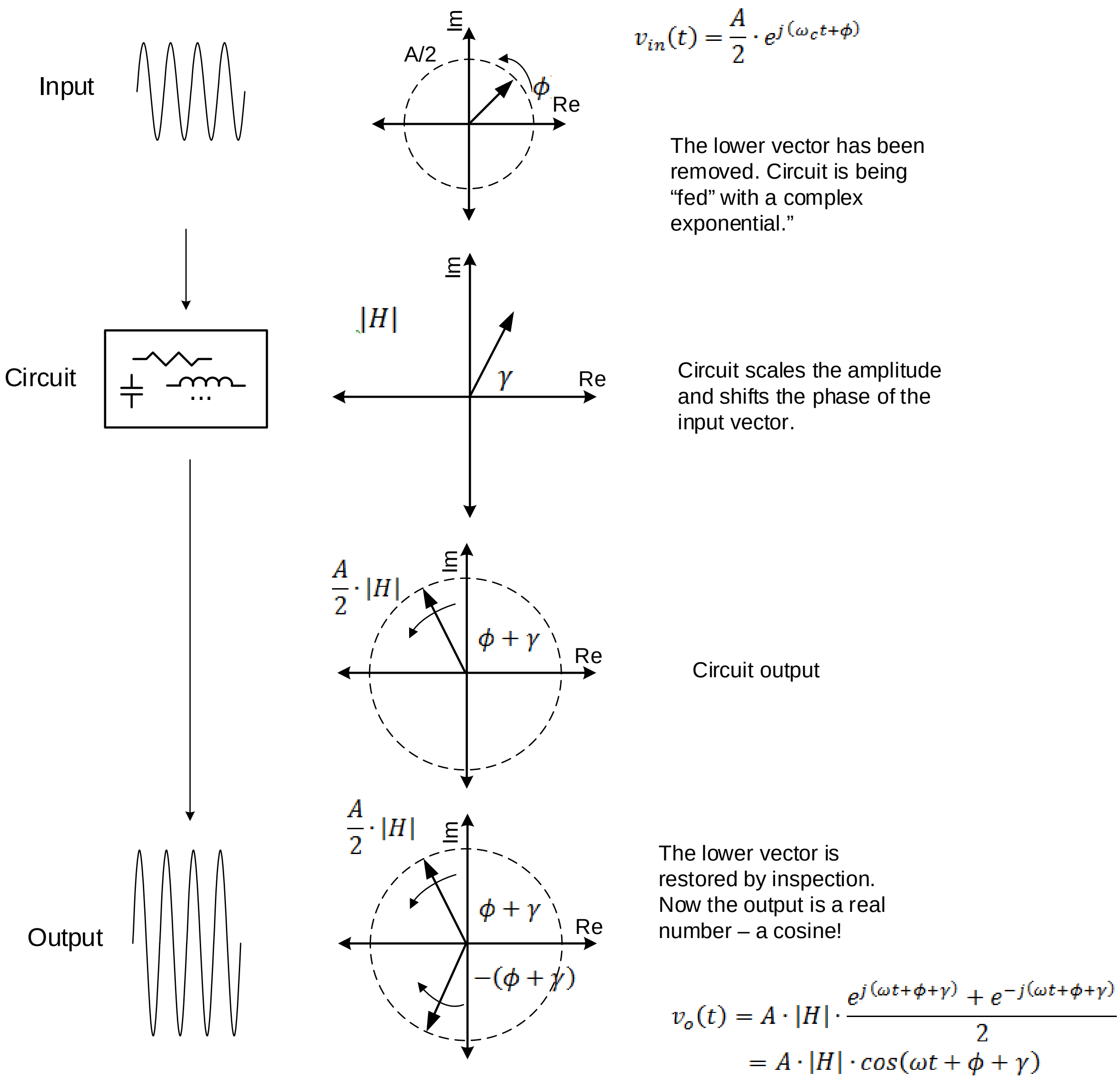


Figure 5 - The process from Figure 4 can be streamlined, resulting in fewer steps

## Circuit Relationships for Sinusoidal Signals

In the previous sections, it was found that complex exponentials can be used to manipulate magnitude scaling and phase shift caused by circuits. It was then found that we can streamline the process by using a complex exponential instead of a cosine or sine. Now the circuit blocks in Figure 4 and Figure 5 are replaced by actual components. Analysis is simplified because the function $e^{j\omega \cdot t}$ has the desirable property that its derivative with respect to time is simply $j\omega \cdot e^{j\omega \cdot t}$.

The current through an inductor is $i_L(t) = e^{j\omega t}$. The voltage across the inductor is:

$$v_L(t) = L \cdot \frac{di}{dt} = L \cdot j\omega \cdot e^{j\omega t} = j\omega L \cdot i_L(t)$$

**Equation 13 – Voltage across an inductor when the current is a sinusoid**

And we can solve for the inductor impedance:

$$\frac{v_L(t)}{i_L(t)} = Z_L = j\omega L$$

**Equation 14 – Impedance of an inductor**

The voltage across a capacitor is $v_C(t) = e^{j\omega t}$. The current through the capacitor is:

$$i_C(t) = C \cdot \frac{dv}{dt} = C \cdot j\omega \cdot e^{j\omega t} = j\omega C \cdot v_C(t)$$

**Equation 15 – Current through a capacitor when the voltage is a sinusoid**

And the impedance of the capacitor is:

$$\frac{v_C(t)}{i_C(t)} = Z_C = \frac{1}{j\omega C}$$

**Equation 16 – Impedance of a capacitor**

So, if the voltages and currents in circuits have the form $e^{j\omega t}$, the inductor impedance can be expressed as $j\omega L$ and the capacitor impedance as $1/j\omega C$. It's an algebraic relationship – not a differential one!

If the input to a circuit is $v_{in}(t) = e^{j\omega t}$, then the voltage and current signals, *s*, at any point, *n*, in the circuit can be expressed as:

$$s_n(t) = A_n e^{j\omega t} \cdot e^{j\phi_n} = v_{in}(t) \cdot A_n \cdot e^{j\phi_n}$$

Equation 17

In other words, the effect of the circuit at any node, *n* is to multiply the complex input by the complex number $A_n e^{j\phi_n}$. The ratio between the circuit output and input signals is called the *transfer function,* $H(j\omega)$.

**Example – Circuit Excited by a Sinusoidal Signal**

Find voltage, $v_o(t)$, in the circuit below.

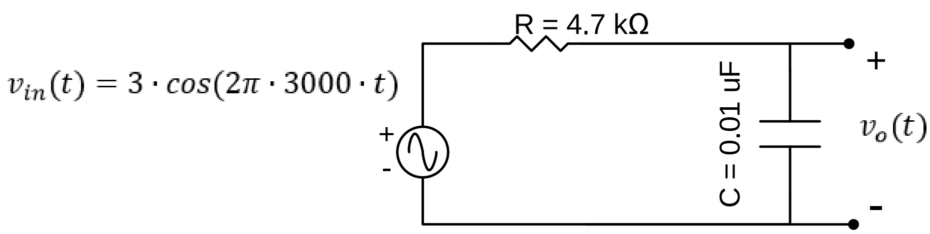


Figure 6 - RC circuit excited by a sinusoidal voltage

Removing the negative frequency vector from the input and using $\omega = 2\pi \cdot 3000$ for the angular frequency gives:

$$v_{in}(t) = \frac{1}{2} \cdot 3 \cdot e^{j2\pi \cdot 3000t} = \frac{3}{2} \cdot e^{j\omega t}$$

Equation 18

The strategy is to compute the complex transfer function:

$$H(\omega) = \frac{v_o(t)}{v_{in}(t)}$$

Equation 19 – The transfer function is the frequency-dependent ratio of the output voltage to the input voltage.

and then compute $v_o(t)$.

$$v_o(t) = \frac{v_o(t)}{v_{in}(t)} \cdot v_{in}(t) = H(\omega) \cdot v_{in}(t)$$

Equation 20

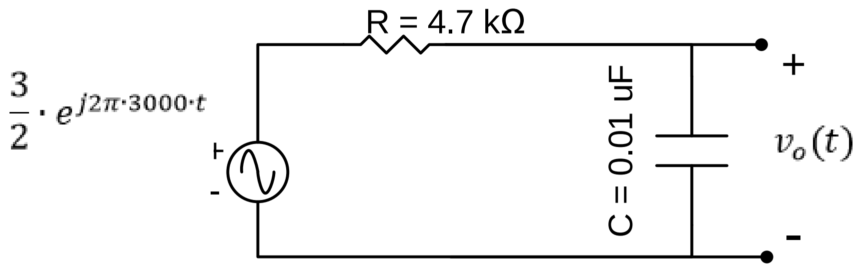


Figure 7- RC circuit excited by a complex exponential voltage

The transfer function, is computed using voltage division.

$$H = \frac{v_o(t)}{v_{in}(t)} = \frac{Z_C}{Z_R + Z_C}$$

Equation 21

Where $Z_R$ and $Z_C$ are the impedances of the resistor and capacitor respectively. The impedance of the resistor is simply its resistance. The impedance of the capacitor is computed using Equation 16:

$$Z_C = \frac{1}{j\omega C}$$

Equation 22 – Impedance of the capacitor

Therefore the transfer function is:

$$H(j\omega) = \frac{Z_C}{Z_C + Z_R} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1}$$

Equation 23

At 3000 Hz, the transfer function evaluates to:

$$H = \frac{1}{j \cdot 2\pi \cdot 3000 \cdot 4700 \cdot 0.01 \times 10^{-6} + 1} = 0.5603 - j0.4964$$

$$= 0.7485 \cdot e^{-j \cdot 0.725}$$

Equation 24 – Transfer function of the RC circuit, Vo/Vin. It is a complex constant at a given frequency.

The positive frequency term (upper vector) of the output signal, $v_o(t)$, is:

$$v_o(t) = v_{in}(t) \cdot H = \frac{3}{2} \cdot e^{j2\pi \cdot 3000t} \cdot 0.7485 \cdot e^{-j \cdot 0.725} = 1.123 \cdot e^{j(2\pi \cdot 3000t - 0.725)}$$

Equation 25

If the negative frequency vector at the input had not been removed, the output would consist of the signal of Equation 25 plus its complex conjugate or

$$v(t) = 1.123 \cdot 2 \cdot \frac{e^{j(2\pi \cdot 3000t - 0.725)} + e^{-j(2\pi \cdot 3000t - 0.725)}}{2}$$

Equation 26

From Equation 4, the output voltage from Equation 26 can be represented as:

$$v(t) = 2.245 \cdot cos(2\pi \cdot 3000 \cdot t - 0.725)$$

Equation 27

And it is seen that the output is a scaled and phase shifted replica of the input.

**The Phasor Method**

The previous example showed that a sinusoidal input can be represented as a complex exponential and used as the input to a circuit. The complex input can then be multiplied by the circuit's transfer function resulting in the complex exponential representing the circuit output. Now we adopt the familiar phasor notation. Since the frequency is the same for any signal in the circuit, phasor notation retains only the magnitude and phase angle. For example:

$$v(t) = V_m \cdot cos(\omega t + \phi_{rad}) \quad \leftrightarrow \quad \boldsymbol{V} = V_m \lfloor \phi_{deg}$$

**Equation 28 – The left side is the time domain; the right side is the phasor domain. Phasors are vectors and typically shown in boldface type. Phasor angles are typically shown in degrees.**

It was also shown that a transfer function at a single frequency is simply a complex scaling constant. For example, in the previous example, the transfer function was the complex ratio of the output voltage to the input voltage.

$$H(j\omega) = \frac{Z_C}{Z_C + Z_R} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1}$$

**Equation 29**

In the previous example, the phasor representation of the input voltage, $v(t) = V_m \cdot cos(\omega t + \phi)$, is

$$v(t) = V_m \cdot cos(\omega t + 0°) \leftrightarrow \boldsymbol{V_{in}} = 3\lfloor 0°\, V$$

**Equation 30**

The phasor representation of the transfer function from Equation 29 is:

$$\boldsymbol{H} = \frac{\boldsymbol{V_o}}{\boldsymbol{V_{in}}} = \frac{1}{j\omega RC + 1} = \ 0.7485\lfloor -41.53° \ \leftrightarrow \ 0.7485 \cdot e^{-j \cdot 0.725}$$

**Equation 31 - Note that phasor notation uses degrees**

The output of the circuit is found by multiplying the input by the transfer function. Since this involves multiplying exponentials, multiply the phasor magnitudes and add the phases to get the answer:

$$\boldsymbol{V_{out}} = \ \boldsymbol{V_{in}} \cdot \boldsymbol{H} = 3\lfloor 0° \ \cdot 0.7485\lfloor -41.54° = 2.246\lfloor -41.54°\, V$$

**Equation 32**

Since $\mathbf{V}_{\mathbf{out}}$ is a phasor, it is converted to a sinusoid by inspection using Equation 28.

$$\boldsymbol{V_{out}} = V_m \lfloor \phi \quad \leftrightarrow \quad V_m \cdot cos(\omega t + \phi) = 2.246 \cdot cos(2\pi \cdot 3000t - 0.725)$$

Equation 33

**Conclusion**

The good news: We can solve steady-state circuit problems with algebra and complex numbers. In the next several lectures, the phasor method will be introduced in more detail. Phasors been used for over a hundred years.

The bad news: The phasor method is so streamlined, and we'll use it so much, it's easy to lose sight of where it came from.

The future: You will encounter complex exponentials in your upper-division courses. Your instructors will expect you to have a working knowledge of them.

Bottom line: Complex exponentials are nothing more than what was presented in this lecture. If you're ever intimidated by complex exponentials, just refer to this lecture. If you can't find it, email me at bdorr@sdsu.edu and I'll send you a copy.