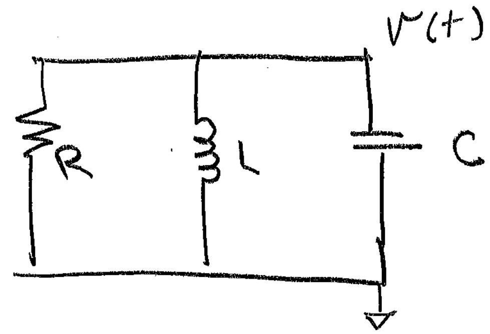
Source-Free Parallel RLC Circuit

Lecture 4

Section 8.4



KCL at top node

$$V_L = L\frac{di}{dt} \quad \text{so} \quad i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau$$

KCL gives

$$\underbrace{\frac{v}{R}}_{\text{Resistor}} + \underbrace{\frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau}_{\text{inductor}} + \underbrace{C\frac{dv}{dt}}_{\text{capacitor}} = 0$$

Take the derivative and divide by C

$$\frac{d^2v}{dt} + \frac{1}{RC}\cdot\frac{dv}{dt} + \frac{1}{LC}\cdot v = 0$$

Assume solutions are $v = Ae^{st}$

so $\frac{d^2v}{dt} = s^2 \cdot Ae^{st}$ and $\frac{dv}{dt} = sAe^{st}$

and our equation becomes

$$s^2 A e^{st} + \frac{1}{RC} s A e^{st} + \frac{1}{LC} A e^{st} = 0$$

factoring out $\underline{Ae^{st}}$ gives the characteristic equation

$$s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0 \qquad \text{EQ } 8.30$$

Compare to characteristic equation for series RLC

$$\underbrace{s^2 + \frac{R}{L} s + \frac{1}{LC} = 0}_{\substack{\text{Series RLC} \\ \text{EQ } 8.7}} \qquad \underbrace{s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0}_{\substack{\text{Parallel RLC} \\ \text{EQ } 8.30}}$$

We find the roots with quadratic formula

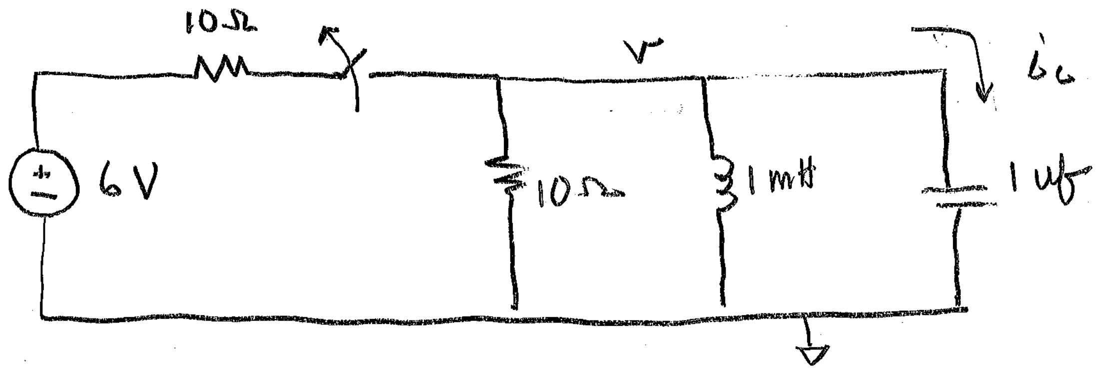
$$s_{1,2} = \underbrace{-\frac{1}{2RC}}_{\alpha} \pm \sqrt{\underbrace{\left(\frac{1}{2RC}\right)^2}_{\alpha^2} - \underbrace{\frac{1}{LC}}_{\omega_o^2}} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

Again compare series and parallel

	α	ω_o
Series	$R/2L$	$1/\sqrt{LC}$
Parallel	$\frac{1}{2RC}$	$1/\sqrt{LC}$

other than this, the analysis is the same

EXAMPLE - See LT SPICE & Excel on Blackboard.



find $v(t)$ for $t > 0$
Strategy ⟶ find $v(t)$ — not always the same!

Initial conditions

inductor is a short at DC, so $I_L(0^-) = I_L(0^+)$
$= 6/10 = 600$ mA

Capacitor voltage = inductor voltage $= 0$ at $t = 0^-$ and $t = 0^+$

Now get dv/dt.

At $t = 0^+$, current in inductor cannot change.

- $V = 0$, so NO current in 10 Ω resistor.
- Therefore all current comes from capacitor. and $i_c(0^+) = -600$ mA

Since $i = c\, dv/dt$, $\frac{dv}{dt} = \frac{i}{c} = \frac{-0.6}{1\,uf} = -600{,}000$ V/s

$$\boxed{\text{So } v(0^+) = 0\text{ V}, \quad \frac{dv}{dt}(0^+) = -600{,}000 \text{ V/s}}$$

initial conditions

Determine the type of response

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 10 \times 1uf} = 50e3$$

$$w_o = \frac{1}{\sqrt{LC}} = 31.6e3$$

Since $\alpha > w_o$, it is over damped and

$$r(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

I use r because it works for Voltage, current, pressure ...

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\frac{dv}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$$

$$V(0^+) = A_1 + A_2 = 0$$

$$\frac{dv(0^+)}{dt} = A_1 S_1 + A_2 S_2 = -600K \ V/s$$

So

$$A_1 + A_2 = 0$$

$$A_1 S_1 + A_2 S_2 = -600K$$

$\longrightarrow$

$$\begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -600K \end{bmatrix}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left[\frac{1}{2RC}\right]^2 - \frac{1}{LC}}$$

$$s_1 = -11.27e3$$
$$s_2 = -88.73e3$$

doesn't matter which is s_1, s_2 provided you use them consistently in the equation

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -11.27e3 & -88.73e3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -600K \end{bmatrix}}$$

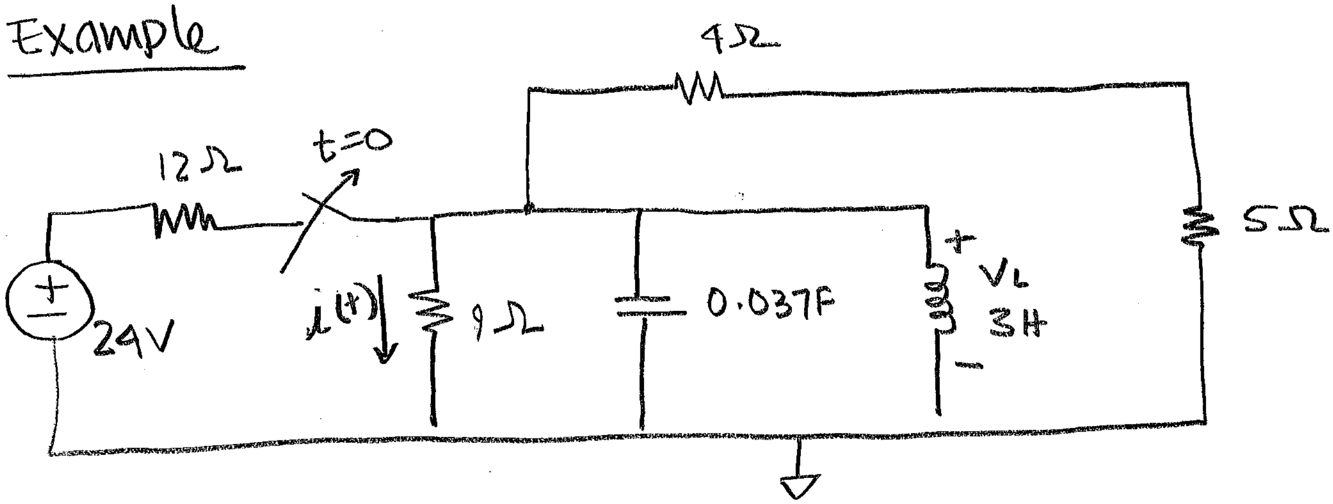
$$A_1 = -7.746, \quad A_2 = 7.746$$

So $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$$v(t) = -7.746\, e^{-11.27e3\, t} + 7.746\, e^{-88.73e3\, t}$$

See LTSpice and Excel implentations on Blackboard.

Example

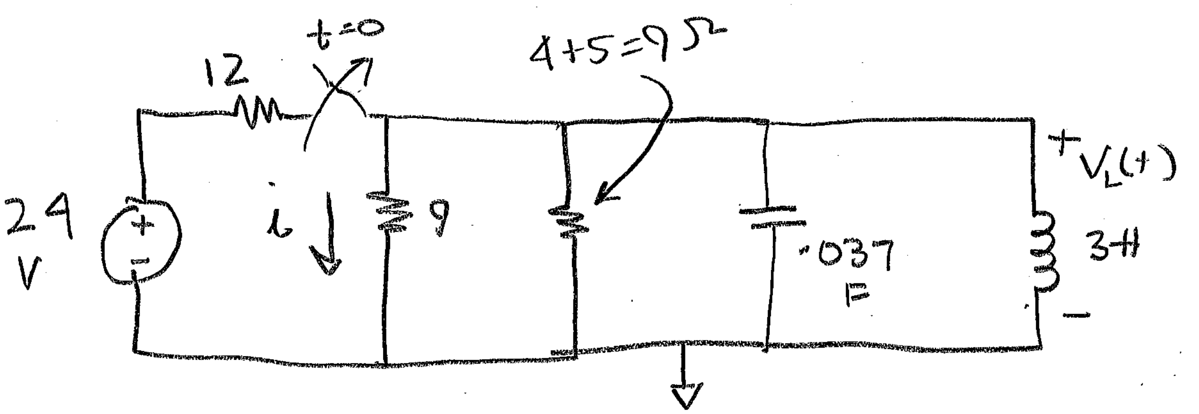


find $i(t)$

Tricky ?

1) The 4 and 5 Ω Resistors make weird topology.

2) Asking for i instead of V for Parallel RLC



Strategy

1) solve for $V(t)$

2) $i(t) = \dfrac{V(t)}{9}$

Since we are solving for $V_L(t)$, we need initial conditions $V_L(0^+)$ and $\frac{dv}{dt}(0^+)$

at $t = 0^-$, $V_L = 0$, $V_C = 0$

$$i_L(0^+) = i_L(0^-) = \frac{24V}{12\Omega} = 2A$$

when switch opens,

1) Capacitor voltage remains at 0. $\longrightarrow V_L(0^+) = 0$
2) No current in resistors
3) inductor current remains 2A
4) Capacitor current is $-2A$

for cap $i = C\frac{dv}{dt}$ so $\frac{dv}{dt} = \frac{i}{C} = \frac{-2}{.037} = -54.05\ V/s$

Summarize initial conditions, $V_L(0^+) = 0$

$$\frac{dV_L}{dt}(0^+) = -54.05\ V/s$$

Next determine the response type

Topology is parallel RLC so

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 4.5 \times .037} = 3$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{3 \times .037}} = 3$$

So it's critically damped.

Use equations from previous lecture

$$V_L(t) = (A_2 + A_1 t)e^{-\alpha t}$$

$$A_2 = V_L(0^+) = 0$$

$$A_1 = \frac{d(V_L)}{dt} + \cancel{\alpha A_2}^{\,0} = -54.05$$

So $\underline{V_L(t) = -54.05\, t\, e^{-3t}}$

Check result before converting to i

We know $V_L(0^+) = 0$

$$-54.05 \times 0 \times e^{-3t} = 0V \checkmark$$

we know $\frac{dv}{dt}(0^+) = -54.05 \text{ V/s}$

$$\frac{d}{dt}\left[-54.05\, t\, e^{-3t}\right]$$

$$= -54.05\, e^{-3t} + 3 \times 54.05\, \cancel{t}\, e^{-3t} \xrightarrow{} 0$$

$$= -54.05 \text{ V/s}$$

we know $V_c(\infty) = 0 \rightarrow$ all energy lost

$$\lim_{t\to\infty}\left[-54.05\, t\, e^{-3t}\right] = \infty \times 0 \quad \text{uh-oh}\ldots$$

$$\lim_{t\to\infty}\left[-54.05\, \frac{t}{e^{+3t}}\right] = \frac{-\infty}{\infty}$$

use L'Hopital's Rule

$$-54.05 \times \frac{1}{3e^{3t}} = -54.05 \times \frac{1}{\infty} = 0 \quad \checkmark$$

Now get final answer by dividing $V(t)$ by $9\ \Omega$

$$i(t) = \frac{V(t)}{9} = -6\, t\, e^{-3t}$$