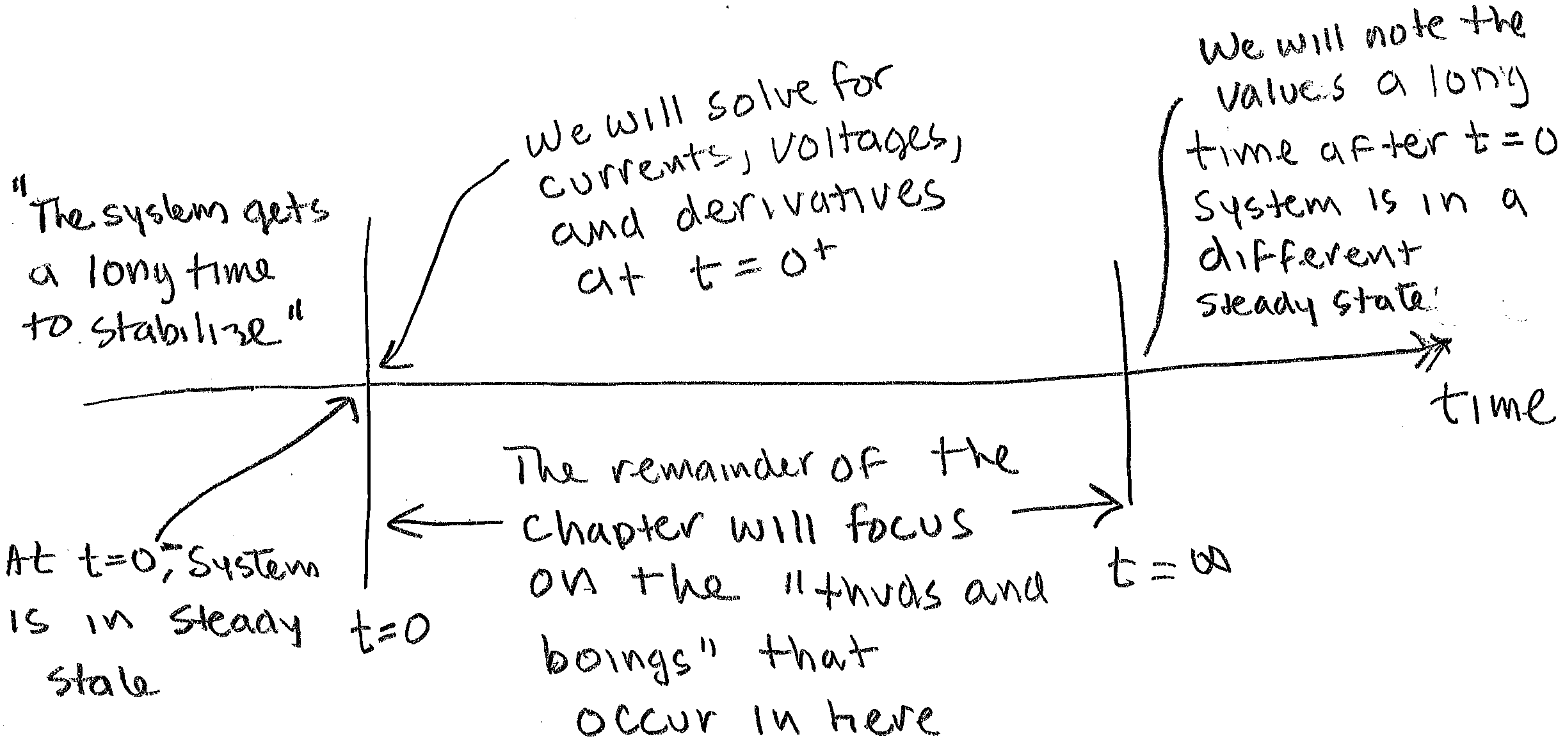
LECTURE #3

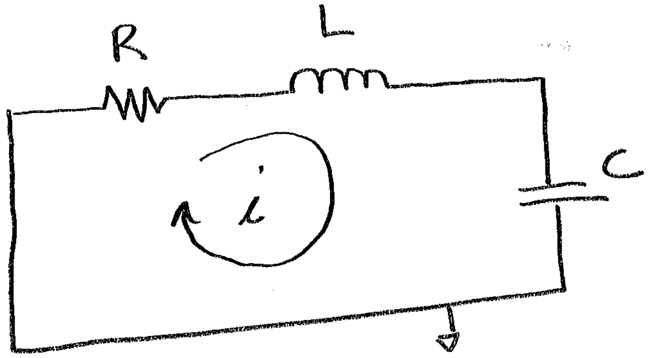
SECTION 8.3

SOURCE FREE RLC CIRCUIT



What Chapter 8 is about

find current as a function of time.



wouldn't this circuit just do nothing?

initial conditions at steady state
- $i_L(0^-)$ → Energy in inductor
- $V_c(0^-)$ → Energy in capacitor

KVL

for capacitor $i_c = C \frac{dv_c}{dt}$

so $V_c = \frac{1}{C}\int_{-\infty}^{t} i_c \, dt$ ← we want to add voltages

so

$$\underbrace{Ri}_{\text{resistor voltage}} + \underbrace{L\frac{di}{dt}}_{\text{inductor voltage}} + \underbrace{\frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau}_{\text{capacitor voltage}} = 0$$ ← EQ 8.3

Eliminate the integral by taking the derivative of the equation.

$$R\frac{di}{dt} + L\frac{d^2i}{dt} + \frac{1}{C}i = 0$$

divide by L and rearrange

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$ ← Second order differential equation. It must have two solutions.

The two solutions have the form $i(t) = Ae^{st}$

We note that $\frac{d^n}{dt}(Ae^{st}) = As^n e^{st}$ ← the "n^{th} derivative"

So $\frac{d^2i}{dt} = s^2e^{st}$ and $\frac{di}{dt} = se^{st}$

so our KVL/differential equation becomes

$$s^2e^{st} + \frac{R}{L}se^{st} + \frac{1}{LC}e^{st} = 0$$

Factoring out e^{st} gives

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$ ← EQ 8.8 in text

"Characteristic Equation"

Value of characteristic equation is that we can solve for s_1, s_2 for our solution

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Solve characteristic equation with the quadratic formula

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = \underbrace{-\alpha \pm \sqrt{\alpha^2 - \omega_0^2}} \quad \longleftarrow \text{EQ 8.10}$$

3 cases to consider:

$\alpha > \omega_0 \rightarrow$ "overdamped" $\longrightarrow$ "thud"

$\alpha = \omega_0 \rightarrow$ "critically damped" $\longrightarrow$ "thump"

$\alpha < \omega_0 \longrightarrow$ "underdamped" $\longrightarrow$ "boing"

<u>Overdamped</u> $(\alpha > \omega_0)$

s_1, s_2 are real, I can plot this

$$r(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \longleftarrow \text{"thud"}$$

"response" voltage current, sound pressure . . .

<u>Critically damped</u> $\longrightarrow$ $\alpha = \omega_0$

$$r(t) = (A_2 + A_1 t)e^{-\alpha t}$$

(see EQ 8.15 to 8.21)

<u>Underdamped</u> $\longrightarrow$ $\alpha < \omega_0$

$$r(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

I can't plot this! because s_1, s_2 are complex.

$$s_{1,2} = -\alpha \pm \sqrt{\underbrace{\alpha^2 - \omega_0^2}_{\text{This is negative}}} = -\alpha \pm \sqrt{-\underbrace{(\omega_0^2 - \alpha^2)}_{\text{This is positive}}}$$

$$s_{1,2} = -\alpha \pm \sqrt{j^2(\omega_0^2 - \alpha^2)} = -\alpha \pm j\omega_d$$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

opposite signs

So $r(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$

or $r(t) = e^{-\alpha t}\left[A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right]$

Euler's identity:

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

so

$$r(t) = e^{-\alpha t}\Big[A_1 \overbrace{\left(\cos(\omega_d t) + j\sin(\omega_d t)\right)}^{\text{Substitution}}$$

$$+ A_2 \underbrace{\left(\cos(\omega_d t) - j\sin(\omega_d t)\right)}_{\text{substitution}}\Big]$$

Collect like terms...

$$r(t) = e^{-\alpha t}\left[(A_1 + A_2)\cos(\omega_d t) + j(A_1 - A_2)\sin(\omega_d t)\right]$$

We will find later that A_1 and A_2 are complex conjugates so...

$$A_1 + A_2 = 2\,\mathrm{Re}\{A_1\} = B_1 \quad \leftarrow \text{complex parts cancel}$$

and

$$j(A_1 - A_2) = j\left[\underbrace{\mathrm{Re}\{A_1\} + j\mathrm{Im}\{A_1\} - \mathrm{Re}\{A_2\}}_{=0} + j\mathrm{Im}\{A_2\}\right]$$

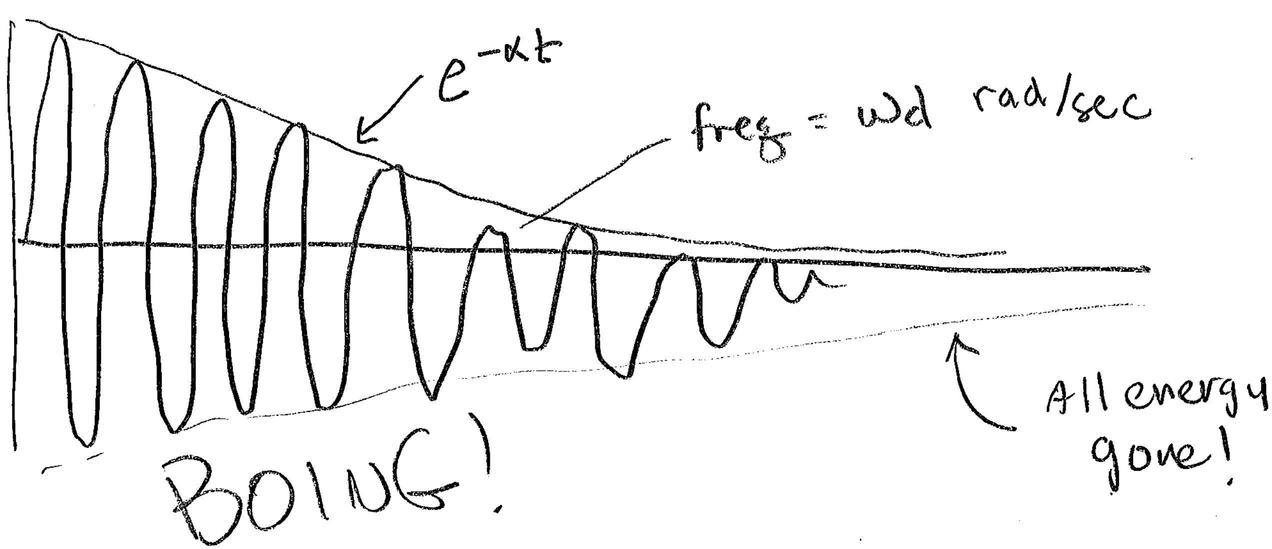
$$= 2j\,\mathrm{Im}\{A_2\}$$

$$= j \cdot \left[2j\,\mathrm{Im}\{A_2\}\right]$$

$$= B_2$$

$$r(t) = e^{-\alpha t}\left[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)\right]$$

EQ 8.26 in text



Summary

Overdamped $\rightarrow r(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \longrightarrow$ Thud

Critically damped $\rightarrow r(t) = (A_2 + A_1 t) e^{st} \longrightarrow$ Thump

Underdamped $\longrightarrow r(t) = e^{-\alpha t}\left[B_1 \cos(w_d t) + B_2 \sin(w_d t)\right]$
"Boing"

But wait, ... what are A_1, A_2, B_1, B_2?
we need those for plotting.

1) They will depend on initial condiions

2) Knowing initial conditions allows us to set up multiple equations in multiple unknowns.

3) Then we get a <u>real</u> expression that we can plot in Excel or see on an osilloscope.

For source-free Series RLC circuit:

$$\alpha = \frac{R}{2L}, \quad w_0 = \frac{1}{\sqrt{LC}}$$

and α and w_0 determine the type of response.

$r(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$, overdamped, $\alpha > w_0$, "Thud"

$r(t) = (A_2 + A_1 t) e^{-\alpha t}$, Critically damped, $\alpha = w_0$, "Thump"

$r(t) = e^{-\alpha t}\left(A_1 \cos(w_d t) + A_2 \sin(w_d t)\right)$,

called these B last time

Underdamped, $\alpha < w_0$, "Boing

Now we learn how to compute constants A_1, A_2

OVERDAMPED

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - w_0^2}$$

$$r(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

So $r(0^+) = A_1 + A_2 \longrightarrow$ Hey that's $r(0^+)$!

$dr/dt = A_1 S_1 e^{s_1 t} + A_2 S_2 e^{s_2 t}$

so $\frac{dr}{dt}(0^+) = A_1 S_1 + A_2 S_2 \longrightarrow$ and this is $\frac{dr(0^+)}{dt}$!

and We have two equations in two unknowns

$$\left.\begin{aligned} A_1 + A_2 &= r(0^+) \\ A_1 s_1 + A_2 s_2 &= \frac{dr}{dt}(0^+) \end{aligned}\right\}$$ this is why we studied initial conditions

Using matrices (the easy way in your calculator)

$$\begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} r(0^+) \\ \frac{dr}{dt}(0^+) \end{bmatrix}$$

or $$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} r(0^+) \\ \frac{dr}{dt}(0^+) \end{bmatrix}$$

<u>Critically damped</u>

$$r(t) = (A_2 + A_1 t)e^{-\alpha t}$$

$$r(0^+) = A_2$$

$$\frac{dr}{dt} = \frac{d}{dt}\left[A_2 e^{-\alpha t} + A_1 t e^{-\alpha t}\right]$$

$$\frac{dr}{dt} = -\alpha A_2 e^{-\alpha t} + A_1 e^{-\alpha t} - A_1 \alpha t e^{-\alpha t}$$

$$\frac{dr}{dt}(0^+) = -\alpha A_2 + A_1$$ so $\longrightarrow$

$$A_2 = r(0^+)$$

$$A_1 = \frac{dr}{dt}(0^+) + \alpha A_2$$

Underdamped

$$r(t) = e^{-\alpha t}\left(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right)$$

$$r(0^+) = A_1$$

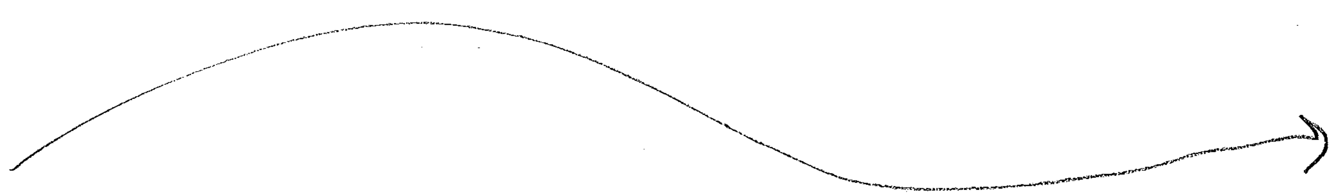
$$\frac{dr}{dt} = -\alpha e^{-\alpha t}\left(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right) +$$

$$e^{-\alpha t}\left(-A_1 \omega_d \cdot \sin(\omega_d t) + A_2 \omega_d \cdot \cos(\omega_d t)\right)$$

$$\frac{dr}{dt}(0^+) = -\alpha A_1 + A_2 \omega_d$$

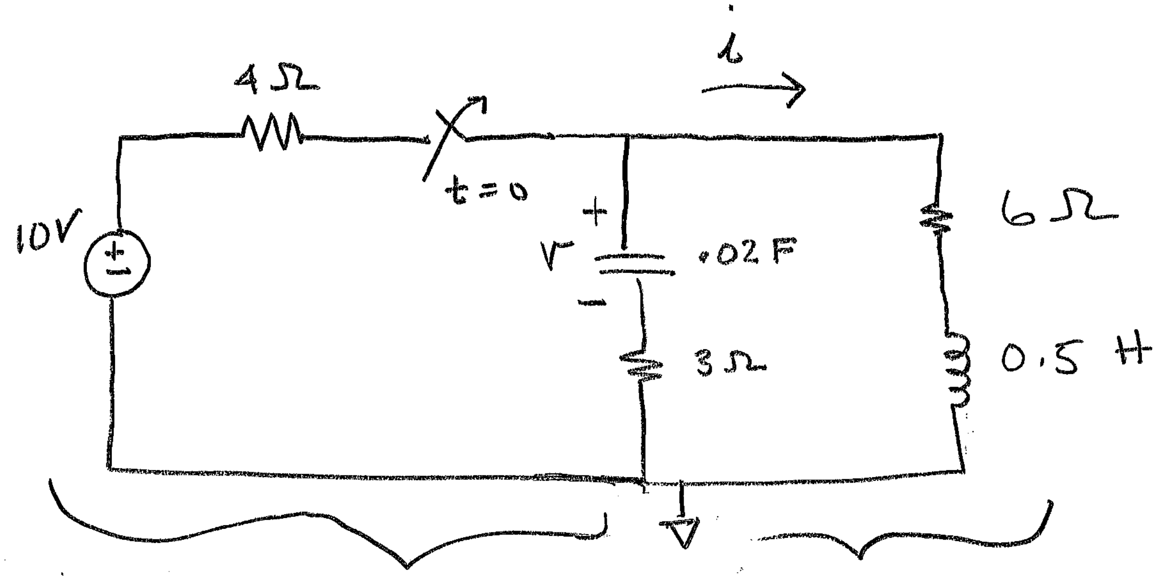
or $\quad A_1 = r(0^+)$

$$A_2 = \frac{\frac{dr}{dt}(0^+) + \alpha A_1}{\omega_d}$$



Example 8.4 Pg 324

Find $i(t)$



This stuff just creates the initial conditions. → after t = 0 we are left with a series RLC circuit.

Strategy

1) Triage the problem. What's going on? What is the topology?
2) Get initial conditions $i(0^+)$ and $\frac{di}{dt}(0^+)$
3) Determine type of response (overdamped, critically damped, underdamped)
4) Compute s_1, s_2 or ω_d (depends on response type)
5) Compute A_1, A_2 and time-domain response
6) Check everything to see if it makes sense

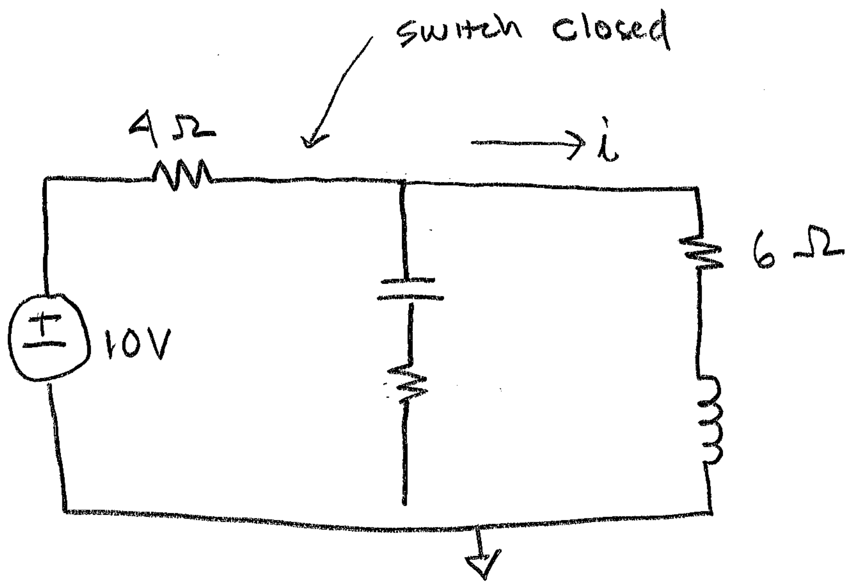
Step 1 - Triage at $t = 0^-$

What is the inductor current ?

what is inductor voltage ?

what is capacitor current ?

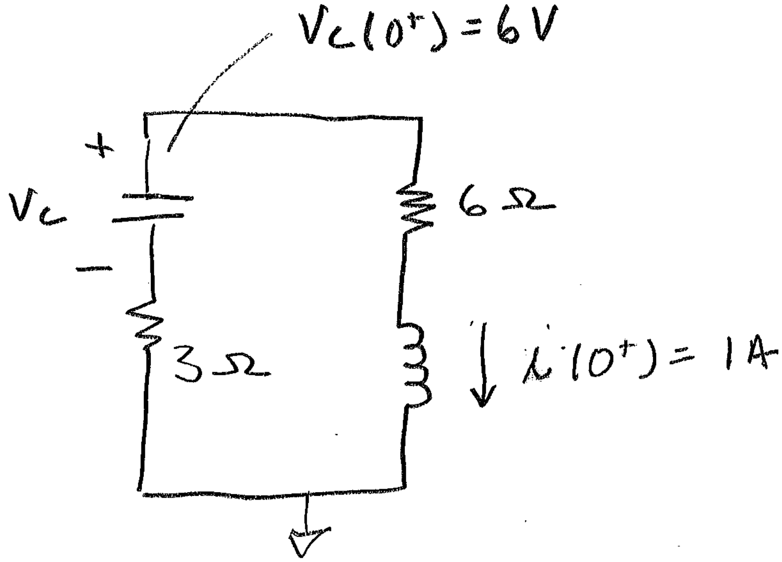
what is capacitor voltage ?



Inductor Current = $\frac{10}{6+4} = 1\,A$, INDUCTOR voltage is zero

Capacitor Voltage = $\frac{6}{6+4} \times 10 = 6V$, Capacitor current is zero

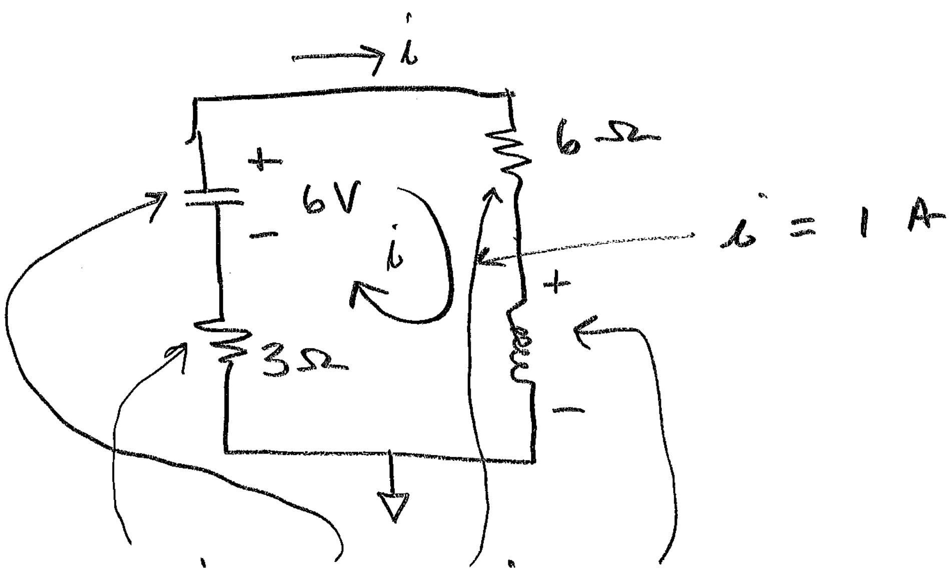
Open the switch at $t=0$



When switch opens, inductor current doesn't change but it now goes through capacitor.

Capacitor voltage can't change quickly, stays at 6 Volts

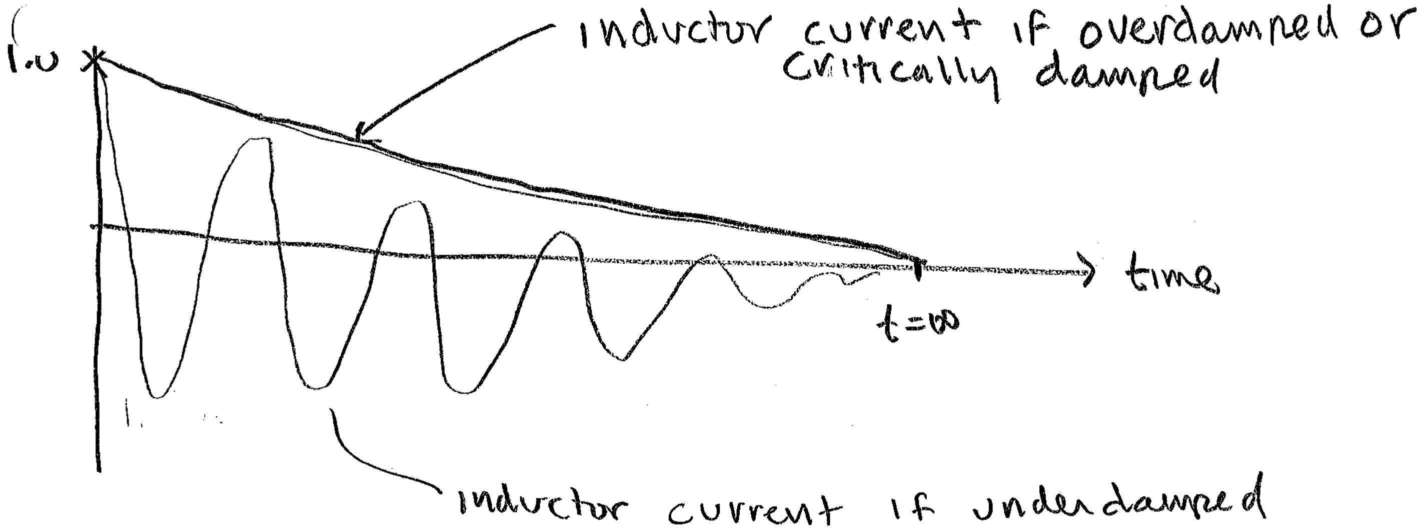
Let's get inductor voltage . . .



$$3 \times i - 6 + 6i + V_{ind} = 0$$

So V_{ind} at $t = 0^+ = 6 - 3i - 6i = 6 - 3 - 6 = -3V$

We know that at $t=\infty$ all voltages and currents in this circuit will be zero



STEP 2 – get initial conditions $i(0^+)$ $\frac{di}{dt}(0^+)$

$$i(0^+) = i_L(0^+) = i_L(0^-) = \frac{10}{6+4} = 1\,A$$

Since $V_L = L\frac{di}{dt}$, $\frac{di}{dt}(0^+) = \frac{V_L(0^+)}{L}$

we found $V_L(0^+)$ during triage, $= -3V$

so $$\frac{di}{dt}(0^+) = \frac{-3}{0.5H} = -6\,\frac{A}{s}$$

Step 3

Determine the type of response

Use EQ 8.11

$$\alpha = \frac{R}{2L} = \frac{9}{1} = 9, \quad w_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.5 \times .02}} = 10 \text{ r/s}$$

$\alpha < w_0$ so underdamped (Pg 321)

Step 4

$w_d = \sqrt{w_0^2 - \alpha^2}$ → top of Pg 323

$$w_d = \sqrt{10^2 - 9^2} = \sqrt{19} = 4.359 \text{ r/s}$$

Step 5

Determine A_1, A_2

Use Equations on Pg 10 of this lecture

$$A_1 = i(0^+) = 1$$

$$A_2 = \frac{\frac{di}{dt}(0^+) + \alpha A_1}{w_d} = \frac{-6 + 9 \times 1}{4.359} = 0.6882$$

Put it all together . . .

$$i(t) = e^{-\alpha t}\left(A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)\right)$$

$$i(t) = e^{-9t}\left[\cos(4.359t) + 0.6882 \cdot \sin(4.359t)\right]$$

Step 6 Check our work

- we triaged $i(0^+) = 1\,A$

substitute in our equation

$$i(0) = e^0\left[\cos(0) + 0\right] = 1.0\,A \checkmark$$

- we triaged $\frac{di}{dt}(0^+) = -6\,A/s$

use our formula for $\frac{di}{dt}$ on pg 10

$$\frac{di}{dt}(0^+) = -\alpha A_1 + A_2 \omega_d$$

$$= -9 \times 1 + 0.6882 \times 4.359 = -6\,\frac{A}{s} \checkmark$$

- we triaged $i(\infty) = 0$

note e^{-9t} as first factor of solution $\checkmark$