LECTURE 25

Circuit Element Models Sec 16.2

These models work for steady state **AND** transient analysis.

Resistor $v(t) = R \cdot i(t)$, $V(s) = R \cdot I(s)$

Inductor $v(t) = L \frac{di}{dt}$

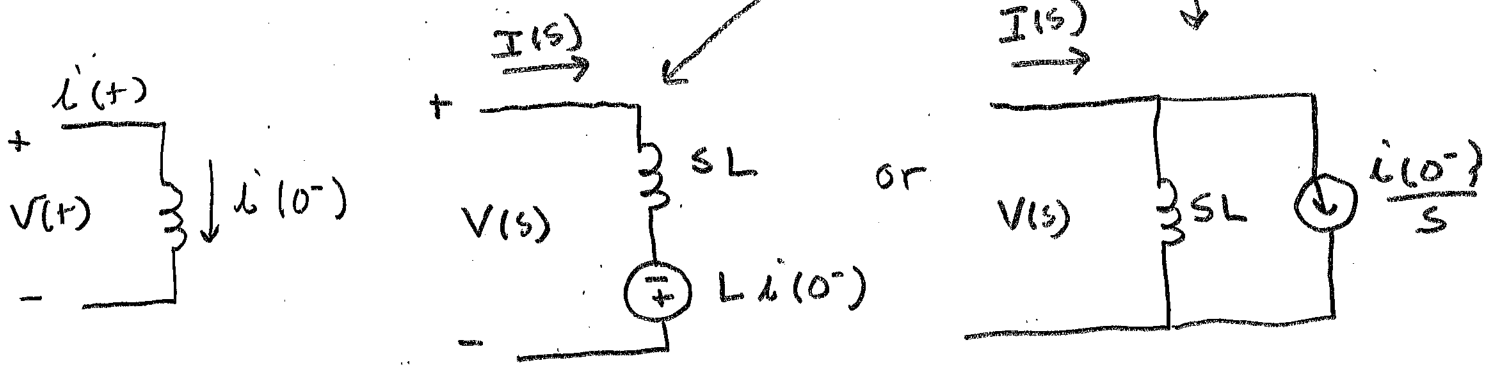
Use time differentiation property

$$V(s) = L\left[s I(s) - i(0^-)\right]$$

Solving for $I(s)$ gives

$$I(s) = \frac{1}{sL} V(s) + \frac{i(0^-)}{s}$$

So Equivalent circuit models are



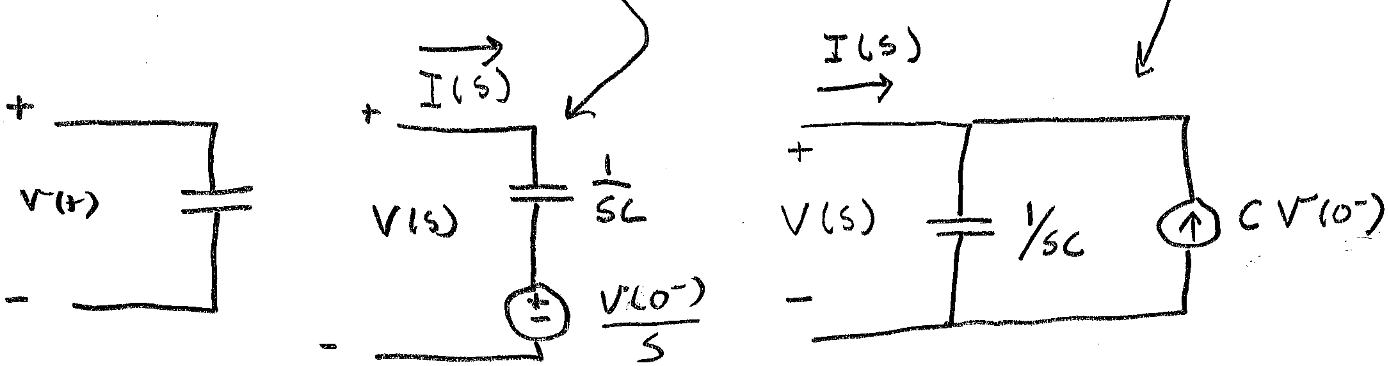
Good for mesh analysis

Good for node Analysis

Capacitor —||— $i(t) = C \frac{dv}{dt}$

so $I(s) = C[s V(s) - v(0^-)]$

or $V(s) = \frac{1}{sC} I(s) + \frac{v(0^-)}{s}$



if initial conditions are zero then.

Resistor $V(s) = R \cdot I(s)$

inductor $V(s) = sL\,I(s)$

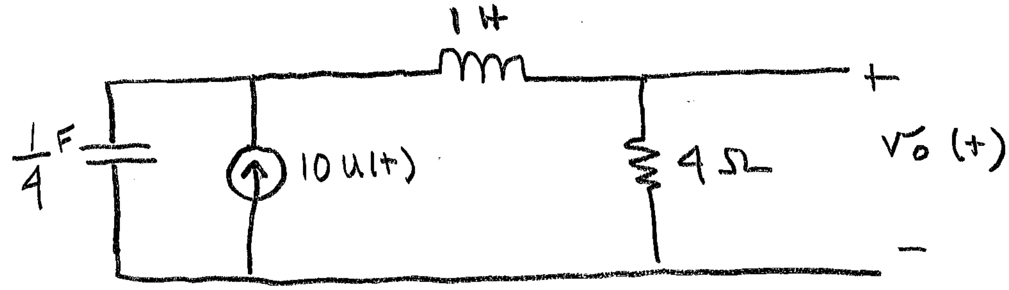
Capacitor $V(s) = \frac{1}{sC} I(s)$

Look familiar?

so "Impedance in the s-domain is the ratio of the voltage transform to the current transform under zero initial conditions" pg 718

"Bridge" between steady state and transient.

Practice Problem 16.1

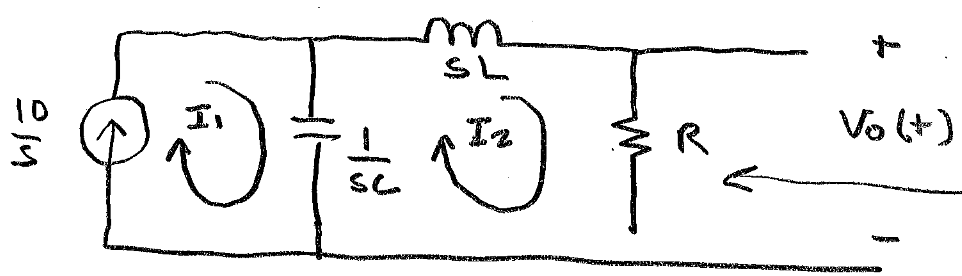


Find $V_o(t)$

All initial conditions are zero

Strategy

1) Transform circuit from time domain to s-domain
2) Determine transfer function $\dfrac{V_o(s)}{I(s)}$
3) Compute $\mathcal{L}^{-1}\{I(s)\cdot H(s)\} = V_o(t)$



dump the numbers

s-domain equivalent

① $I_1 = \dfrac{10}{s}$ ③ $V_o = I_2 R$ or $I_2 = \dfrac{V_o}{R}$

② $\dfrac{1}{sC}(I_2 - I_1) + sLI_2 + I_2R = 0$

② using ① $\dfrac{1}{sC}\left(I_2 - \dfrac{10}{s}\right) + sLI_2 + I_2R = 0$

we will just solve for $V_o(s)$

② using ③ $\dfrac{1}{sC}\left(\dfrac{V_o}{R} - \dfrac{10}{s}\right) + sL\dfrac{V_o}{R} + \dfrac{V_o}{R}\times R = 0$

(2) $$\frac{V_o}{SCR} - \frac{10}{S^2C} + \frac{SLV_o}{R} + V_o = 0$$

$$V_o\left[\frac{1}{SCR} + \frac{SL}{R} + 1\right] = \frac{10}{S^2C}$$

$$V_o = \frac{10}{S^2C} \times \frac{1}{\frac{1}{SCR} + \frac{SL}{R} + 1}$$

$$V_o = 10 \times \frac{1}{\frac{S}{R} + \frac{S^3LC}{R} + S^2C}$$

$$V_o = 10 \times \frac{1}{S\left(\frac{S^2LC}{R} + SC + \frac{1}{R}\right)}$$

$$V_o = \frac{10}{LC/R} \times \frac{1}{S\underbrace{\left(S^2 + \frac{SR}{L} + \frac{1}{LC}\right)}}$$

why is "characteristic equation" the same as Eq 8.8?

is this two simple poles or two complex poles "Thud or boing"?

$$\left.S^2 + S \times \frac{4}{1} + \frac{1}{1 \times \frac{1}{4}}\right\} = S^2 + 4S + 4 = (S+2)^2$$

It's a repeated pole

$$V_o(s) = 160 \times \frac{1}{s(s+2)^2}$$ $\longrightarrow$ Get inverse Laplace using Properties to study for final!

That was a lot of work. Let's check with inital and final value theorems

$$f(0) = \lim_{s\to\infty}\left\{sF(s)\right\} = \lim_{s\to\infty}\left\{160 \times \frac{1}{(s+2)^2}\right\} = 0\ V$$

$$f(\infty) = \lim_{s\to 0}\left\{\frac{160}{(s+2)^2}\right\} = 40\ V$$

Does this agree with the circuit?

Partial fractions

strategy:

1) use Eq 15.54 (Guarantee)

$$\frac{160}{s(s+2)^2} = \frac{K_1}{(s+2)^2} + \frac{K_2}{s+2} + \frac{K_3}{s}$$

2) Use method of Algebra to get K_1, K_2, K_3

$$\frac{160}{s(s+2)^2} = \frac{K_1 s + K_2 s(s+2) + K_3(s+2)^2}{s(s+2)^2}$$

Equate numerators

$$160 = K_1 s + K_2 s^2 + 2K_2 s + K_3 s^2 + 4K_3 s + 4K_3$$

$$160 = s^2(K_2+K_3) + s(K_1+2K_2+4K_3) + 4K_3$$

so $$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 160 \end{bmatrix}$$

and $K_1 = -80$
$K_2 = -40$
$K_3 = 40$

so $$H(s) = \frac{160}{s(s+2)^2} = \frac{-80}{(s+2)^2} - \frac{40}{s+2} + \frac{40}{s}$$

$$h(t) = \left[-80te^{-2t} - 40e^{-2t} + 40\right]u(t)$$

$$= 40\left[1 - e^{-2t} - 2te^{-2t}\right]u(t)$$

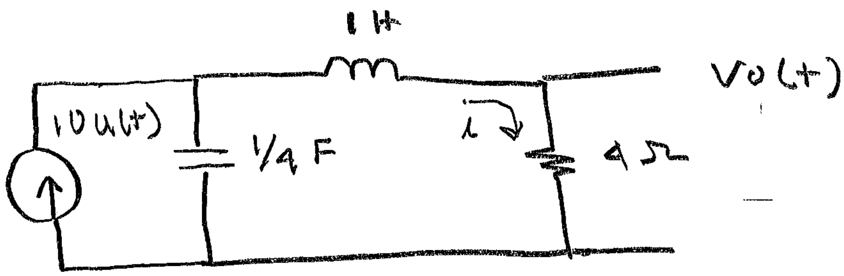
Check it!

$f(0) = 40(1-1-0) = 0$ ✓

$f(\infty) = 40[1-0-0]$ ✓

Practice problem 16.1 Find $V_o(t)$

... The chapter 8 way



Series RLC

$\alpha = \frac{R}{2L} = \frac{4}{2} = 2$

$\omega_0 = \frac{1}{\sqrt{LC}} = 2$

$V_o(\infty) = V_s = ?$

$V_o(0^+) = ?$

$\frac{dv_o}{dt}(0^+) = ?$

since $\alpha = \omega_0$ it is critically damped

General form

$$r(t) = R_s + (A_1 + A_2 t)e^{-\alpha t}$$

or $v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t}$

$v(0^+) = 0$ why? $V_s = V_\infty = 40$ why?

$A_1 = v(0^+) - V_s = 0 - 40 = -40$

$A_2 = \frac{dv}{dt}(0^+) + \alpha A_1$

$\hookrightarrow = R\frac{di}{dt} = 4 \times 0 = 0$

see "chapter 8 - useful equations" under study materials

so

$A_2 = 0 + \alpha A_1 = 2 \times -40 = -80$

$$\boxed{V(t) = 40 + (-40 - 80t)e^{-2t}}$$

Same as we got with Laplace xform

Problem 16.2

$$\frac{d^2v}{dt} + 5\frac{dv}{dt} + 4v = 0$$

Given $v(0) = 0$, $\frac{dv(0)}{dt} = 5\ V/s$, find $v(t)$

Convert to Laplace domain

$$\underbrace{\frac{d^2v}{dt}}_{\text{time differentiation}} + \underbrace{5\frac{dv}{dt}}_{\text{time differentiation}} + 4V$$

$$s^2V(s) - s\cdot \cancel{V(0^-)}^{0} - \underbrace{V'(0^-)}_{=5} + 5\left(sV(s) - \cancel{V(0^-)}^{0}\right) + 4V(s) = 0$$

$$s^2V + 5sV + 4V - 5 = 0$$

$$V(s^2 + 5s + 4) = 5$$

or $$V = \frac{5}{s^2+5s+4} = \frac{5}{(s+4)(s+1)}$$

use method of algebra

$$\frac{5}{s^2+5s+4} = \frac{A_1}{s+4} + \frac{A_2}{s+1} = \frac{A_1(s+1) + A_2(s+4)}{s^2+5s+4}$$

so $$\frac{5}{s^2+5s+4} = \frac{A_1s + A_1 + A_2s + 4A_2}{s^2+5s+4}$$

$$5 = s(A_1 + A_2) + A_1 + 4A_2$$

so $\quad A_1 + A_2 = 0 \qquad A_1 = -5/3$

$\quad\;\; A_1 + 4A_2 = 5 \qquad A_2 = 5/3$

so $v(t) = -\frac{5}{3}e^{-4t} + \frac{5}{3}e^{-t}$

check using $v(0^-)$ and $\frac{dv}{dt}(0^-)$

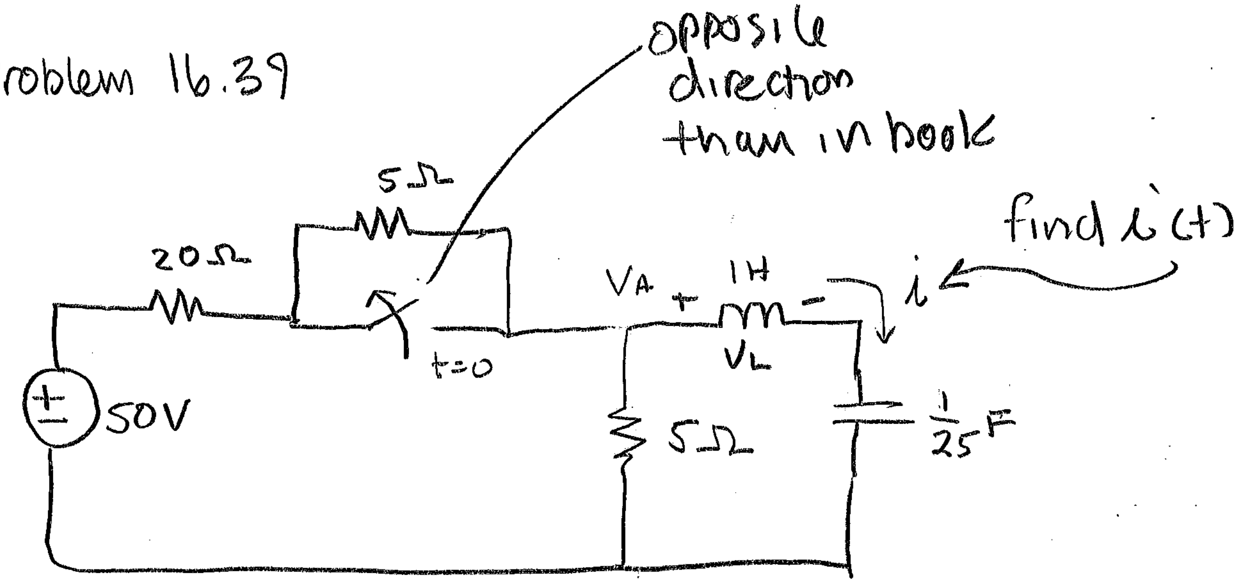
$$V(0) = -\frac{5}{3} + \frac{5}{3} = 0 \quad ✓$$

$$\frac{dv}{dt} = \frac{4 \times 5}{3} e^{-4t} - \frac{5}{3} e^{-t}$$

$$\frac{dv}{dt}(0) = \frac{20 - 5}{3} = \frac{15}{3} = 5 \quad ✓$$

No need to check answer. I'm comfortable with this!

Problem 16.39



Triage

What are V_A and i at $t=0^-$?

What is i at $t=0^+$?

What is V_A at $t=0^+$

What is di/dt at $t=0^+$?

$V_A(0^-) = \frac{5}{25} \times 50 = 10V$

$V_A(0^+) = \frac{5}{30} \times 50 = 8.33V$

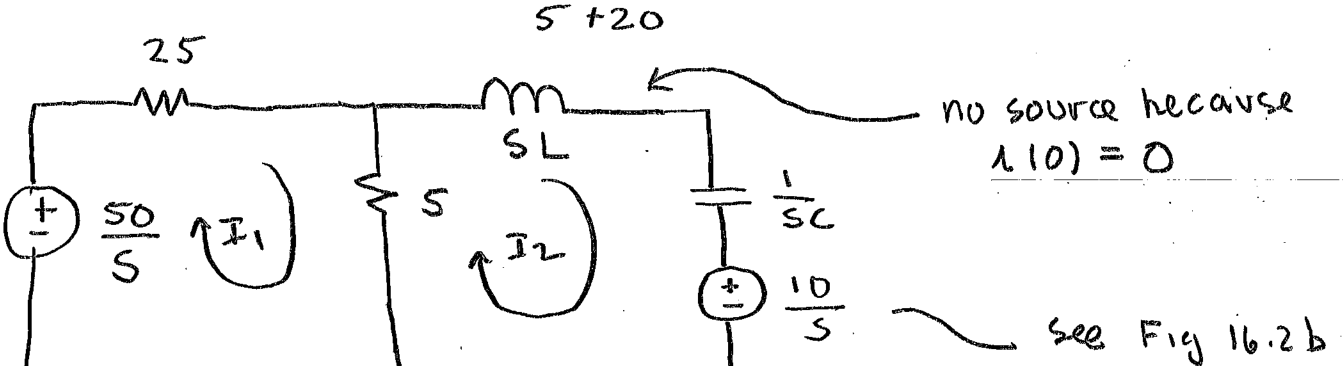
so $V_L(0^+) = 8.33 - 10 = -1.67V$

$\frac{di_L}{dt} = \frac{-1.67}{1H} = -1.67 A/s$

What kind of circuit is this?

Are differential equations or Laplace the best method?

$V_C(0^-) = V_C(0^+) = 50 \times \frac{5}{5+20} = 10\ V$



Note that we don't need derivatives ($\frac{dV_C(0^+)}{dt}, \ldots$) with Laplace, Just initial conditions

① $-\frac{50}{s} + 25I_1 + 5(I_1 - I_2) = 0$

② $5(I_2 - I_1) + sLI_2 + \frac{I_2}{sC} + \frac{10}{s} = 0$

① $-\frac{50}{s} + 25I_1 + 5I_1 - 5I_2 = 0$

① $5I_2 = 25I_1 + 5I_1 - \frac{50}{s}$

① $I_2 = 6I_1 - \frac{10}{s}$

② $5I_2 - 5I_1 + sLI_2 + \frac{I_2}{sC} + \frac{10}{s} = 0$

$I_2\left(5 + sL + \frac{1}{sC}\right) - 5I_1 = -\frac{10}{s}$

② $\left(6I_1 - \frac{10}{s}\right)\left(5 + sL + \frac{1}{sC}\right) - 5I_1 = -\frac{10}{s}$

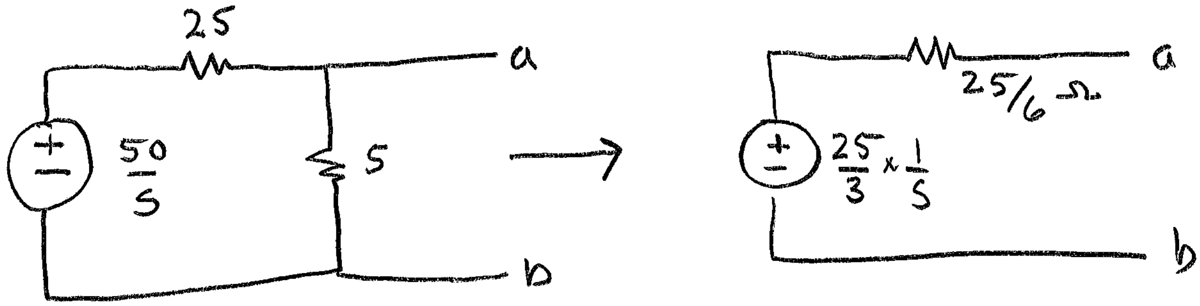
② $\underbrace{30I_1 + 6I_1sL + \frac{6I_1}{sC} - \frac{50}{s} - 10L - \frac{10}{s^2C} - 5I_1 = -\frac{10}{s}}$

Yuk....

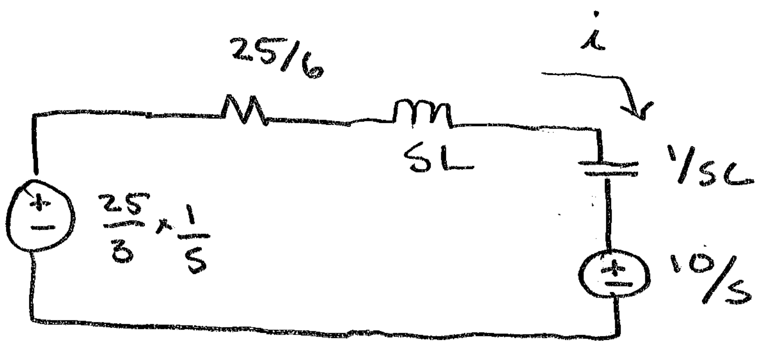
Note Thevenin transformation easily converts this to a single loop....

Re-start ⟶

PR 16.39 CONT



$$V_{Th} = \frac{50}{s} \times \frac{5}{25+5} = \frac{25}{3} \times \frac{1}{s}\ , \quad R_{Th} = \frac{5 \times 25}{5+25} = \frac{125}{30} = \frac{25}{6}$$



$$-\frac{25}{3} \times \frac{1}{s} + \frac{25}{6} I + sLI + \frac{I}{sC} + \frac{10}{s} = 0$$

$$I\left(\underbrace{\frac{25}{6}}_{4.17} + sL + \frac{1}{sC}\right) = \left(\underbrace{\frac{25}{3}}_{8.33} - 10\right) \times \frac{1}{s}$$

$$I = \frac{-1.667}{s} \times \frac{1}{4.17 + sL + 1/sC}$$

$$I = \frac{-1.667}{s^2 L + 4.17 s + 1/C}$$

$$I = \frac{-1.667}{s^2 + 4.17s + 25} \qquad \sqrt{b^2 - 4ac} \text{ is neg}$$

$$I(s) = \frac{-1.667}{s^2 + 4.17s + 25} = \frac{A_1 s + A_2}{s^2 + \underset{\uparrow a}{4.17} s + \underset{\uparrow b}{25}} \leftarrow \text{GUARANTEE}$$

Equate coeffs, $A_1 = 0$, $A_2 = -1.667$

Use formulas we derived in class

$$\alpha = \frac{a}{2} = \frac{4.17}{2} = 2.08, \quad B = \sqrt{b - \frac{a^2}{4}} = \sqrt{25 - \frac{4.17^2}{4}}$$

$$= 4.54$$

and $B_1 = \dfrac{A_2 - A_1\alpha}{B} = \dfrac{-1.667 - 0}{4.54} = -0.3672$

EQ 15.65

So $i(t) = -0.3672\, e^{-2t} \sin(4.54t)\, u(t) = i(t)$

disagrees with book because our switch opened at $t=0$

Check against circuit.

$i(0) = -0.3672 e^0 \sin(0) = 0$

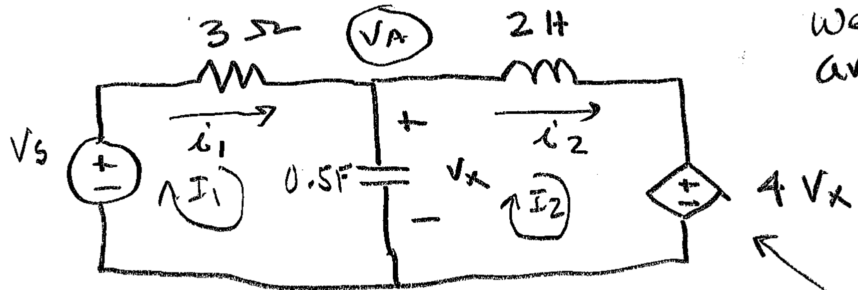
$$\frac{di}{dt} = -0.3672 \left[-2e^{-2t} \underbrace{\cancel{\sin(4.54t)}}_{0} + \underbrace{\cancel{e^{-2t}}}_{1} \times 4.54 \times \underbrace{\cancel{\cos(4.54t)}}_{1} \right]$$

$$-0.3672 \times 4.54 = \underline{-1.667\ A/s}$$

which agrees with result on pg 10

PR 16.79

Find V_A/V_S



$\rightarrow \quad V_x = (I_1 - I_2) \times \frac{1}{sC}$

We must assume initial cond are 0.

① $-V_s + 3I_1 + \frac{1}{sC}(I_1 - I_2) = 0$

② $(I_2 - I_1)\frac{1}{sC} + sLI_2 + 4(I_1 - I_2) \times \frac{1}{sC}$

mesh approach

Node will give me a single equation

$$\frac{V_s - V_A}{3} - \frac{V_A}{1/sC} + \frac{4V_A - V_A}{sL} = 0$$

dep source

$$V_A\left(-\frac{1}{3} - sC + \frac{3}{sL}\right) = -\frac{V_s}{3}$$

$$V_A = -\frac{V_s}{3} \times \frac{1}{\frac{3}{sL} - sC - \frac{1}{3}}$$

$$\frac{V_A}{V_s} = \frac{1}{3\left(\frac{1}{3} + sC - \frac{3}{sL}\right)} = \frac{1}{1 + 3sC - \frac{9}{sL}}$$

$$\frac{V_A}{V_s} = \frac{s}{3s^2C + s - 9/L} = \frac{1}{3C} \times \frac{s}{s^2 + \frac{s}{3C} - \frac{9}{3LC}}$$

so $$\frac{V_A}{V_S} = \frac{2}{3}\left[\frac{s}{s^2 + 2/3\, s - 3}\right]$$

Say we input a unit step. what would the circuit do?

Now let's look at the math

$$V_A = \frac{1}{s} \times \frac{2}{3} \times \frac{s}{s^2 + 2/3\, s - 3}$$

$$= \frac{2}{3} \times \frac{1}{(s - 1.42888)(s + 2.1)}$$

pole is in Right half plane!

$$= \frac{K_1}{s - 1.43} + \frac{K_2}{s + 2.1}$$

and $$f(t) = K_1 e^{1.43t} + K_2 e^{-2.1t}$$

Does this justify what you obsérve from the circuit?

Say the dependent source was $-4V_x$ instead of $4V_x$, then

$$\frac{V_S - V_A}{R} - \frac{V_A}{1/SC} + \frac{-4V_A - V_A}{SL} = 0$$

or $$\frac{V_A}{V_S} = \frac{1}{RC} \times \frac{S}{S^2 + \frac{1}{RC}S + \frac{5}{LC}}$$

$$\frac{V_A}{V_S} = \frac{2}{3} \times \frac{S}{S^2 + 0.667S + 5}$$

What would the circuit do if we input a unit step?

Now the Math...

$$V_A(s) = \frac{1}{S} \times \frac{2}{3} \times \frac{S}{(S^2 + 0.667S + 5)}$$

Poles are $P_1, P_2 = \frac{-0.67 \pm \sqrt{0.67^2 - 20}}{2}$

real part is negative so stable.

Final value

$$f(\infty) = \lim_{S \to 0}(S F(S)) = \lim_{S \to 0}\left\{\frac{2}{3} \times \frac{S}{S^2 + 0.667S + 5}\right\}$$

$$= 0!$$

See LT SPICE file posted on Blackboard