Lecture #23 - Complex Poles - Convolution

Transfer functions can have three kinds of poles

Simple poles $\rightarrow H(s) = \dfrac{n(s)}{(s+p_1)(s+p_2)}$

repeated Poles $\rightarrow H(s) = \dfrac{n(s)}{(s+p)^2}$

we covered these in the last lecture

complex poles

$$F(s) = \frac{n(s)}{s^2+as+b}$$

This lecture

Here's the guarantee if there is a complex pole

$$F(s) = \underbrace{\frac{A_1 s + A_2}{s^2+as+b}} + F_1(s) \qquad \text{EQ 15.61}$$

Remaining part that does not have complex pole pair.

we convert to:

IN LAPLACE TABLE

$$F(s) = \frac{A_1(s+\alpha)}{(s+\alpha)^2+\beta^2} + \frac{B_1\beta}{(s+\alpha)^2+\beta^2} + F_1(s) \qquad \text{EQ 15.64}$$

and then the inverse laplace xform is easy

EQ 15.65 $$f(t) = \left[A_1 e^{-\alpha t}\cos(\beta t) + B_1 e^{-\alpha t}\sin(\beta t)\right]u(t) + f_1(t)$$

- First we solve for A_1 and A_2 → use method of algebra
- Next we solve for α and β
- Then we solve for B_1

→ solve for α and β

we want $s^2 + as + b = (s+\alpha)^2 + \beta^2$

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2$$

$$2\alpha = a \quad \text{or} \quad \alpha = \frac{a}{2}$$

$$b = \alpha^2 + \beta^2$$

$$\text{so} \quad \beta = \sqrt{b - \alpha^2} = \sqrt{b - \frac{a^2}{4}} = \beta$$

$$\boxed{\text{so} \quad \alpha = \frac{a}{2}, \quad \beta = \sqrt{b - \frac{a^2}{4}}}$$ ← not in book

Now solve for B_1

$$A_1(s+\alpha) + B_1 \cdot \beta = A_1 s + A_2$$

← Equate numerator of 15.61 to numerator of 15.64.

$$\cancel{A_1 s} + A_1 \alpha + B_1 \beta = \cancel{A_1 s} + A_2$$

$$\boxed{B_1 = \frac{A_2 - A_1 \alpha}{\beta}}$$ ← not in book

Example 15.11 Pg 695

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)} \qquad \frac{-8 \pm \sqrt{8^2 - 4 \times 25}}{2}$$

Why do I do this?

$$H(s) = \frac{A_1 s + A_2}{s^2+8s+25} + \frac{A_3}{s+3}$$ ← EQ 15.61

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)} = \frac{(A_1 s + A_2)(s+3) + A_3(s^2+8s+25)}{(s+3)(s^2+8s+25)}$$

Equate numerators

$$20 = A_1 s^2 + A_2 s + 3A_1 s + 3A_2 + A_3 s^2 + 8A_3 s + 25A_3$$

$$20 = s^2(A_1 + A_3) + s(A_2 + 3A_1 + 8A_3) + 3A_2 + 25A_3$$

so

$$A_1 + 0A_2 + A_3 = 0$$
$$3A_1 + A_2 + 8A_3 = 0$$
$$0A_1 + 3A_2 + 25A_3 = 20$$

or

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 8 \\ 0 & 3 & 25 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -10 \\ 2 \end{bmatrix}$$

so

$$\frac{20}{(s+3)(s^2+8s+25)} = \frac{-2s-10}{s^2+8s+25} + \frac{2}{s+3}$$

$s^2 + as + b \Longrightarrow a = 8,\ b = 25$

and $A_1 = -2,\ A_2 = -10$

use our equations derived on page 2

$$\alpha = \frac{a}{2} = 4, \quad \beta = \sqrt{b - \frac{a^2}{4}} = \sqrt{25 - \frac{64}{4}} = 3$$

$$\text{and } B_1 = \frac{A_2 - A_1\alpha}{\beta} = \frac{-10 + 2 \times 4}{3} = -\frac{2}{3}$$

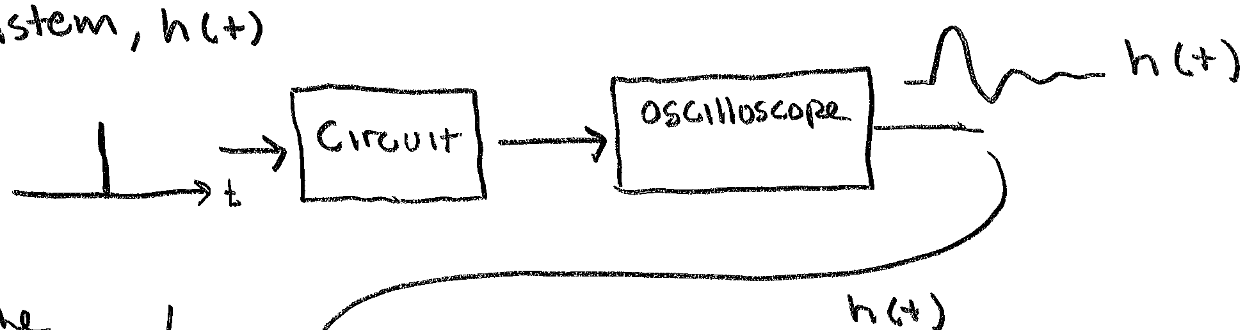
and EQ 15.65 gives

$$f(t) = \left[A_1 e^{-\alpha t}\cos(\beta t) + B_1 e^{-\alpha t}\sin(\beta t)\right]u(t) + f_1(t)$$

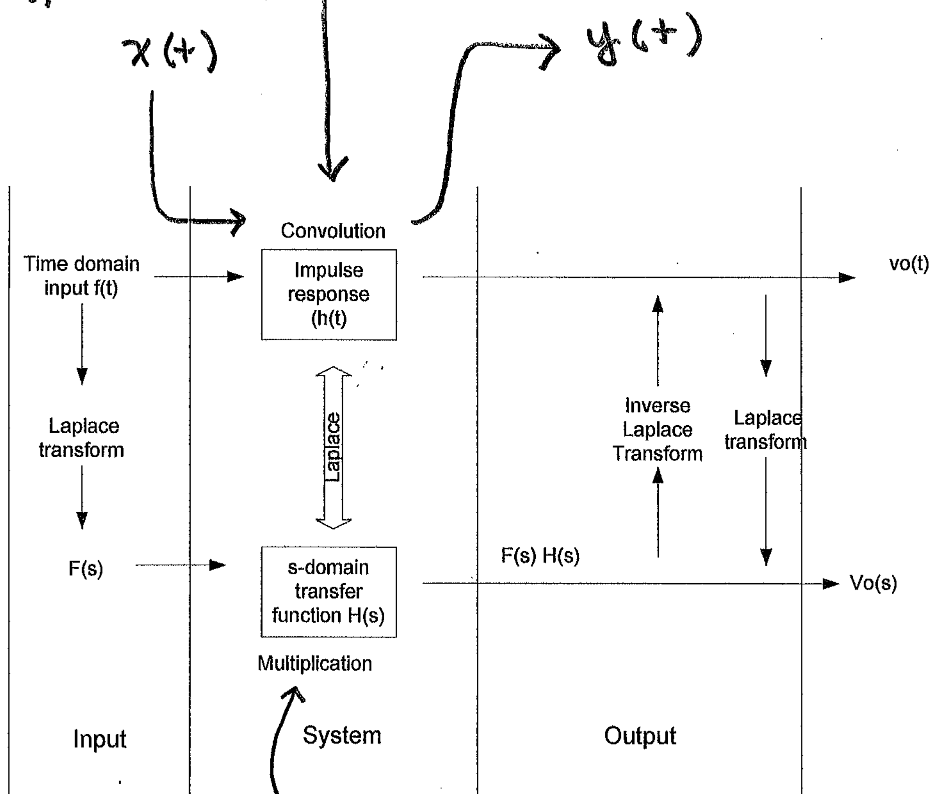
$$f(t) = \left[-2e^{-4t}\cos(3t) - \frac{2}{3}e^{-4t}\sin(3t) + 2e^{-3t}\right]u(t)$$

SECTION 15.5 Convolution integral

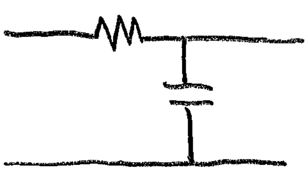
Allows us to determine the response of a system to an exitation $x(t)$, knowing the impulse response of the system, $h(t)$



Measure the impulse response!



or compute impulse response...



$$H(s) = \frac{1}{RC} \cdot \frac{1}{s + 1/RC}, \quad \mathcal{L}^{-1}(H(s)) = \frac{1}{RC} e^{-t/RC} = h(t)$$

convolution integral – EQ 15.66, 15.68

$$y(t) = \underbrace{\int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda}$$

λ integrates out leaving

$$y(t) = x(t) * h(t)$$

↑ special operator for convolution

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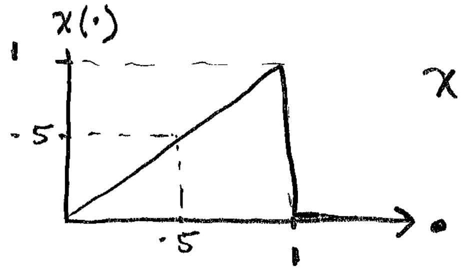
time reversing and shifting functions → insight

- Say we know what $x(t)$ looks like.
What does $x(t-\lambda)$ look like on the λ axis?
(we integrate on the λ axis)

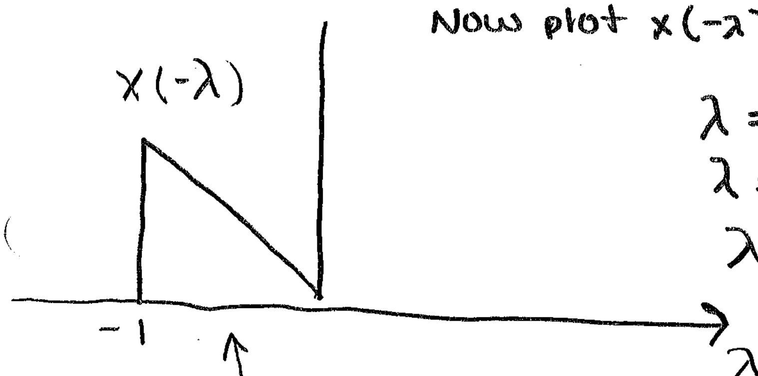


$x(\cdot)$

$x(0) = 0$
$x(0.5) = 0.5$
$x(1) = 1$

x is a function of whatever is in the parenthesis

Now plot $x(-\lambda)$ on the λ axis



$\lambda = 0, \quad x(-\lambda) = x(0) = 0$
$\lambda = -0.5, \quad x(-\lambda) = x(0.5) = 0.5$
$\lambda = -1, \quad x(-\lambda) = x(1) = 1$

now let's plot $x(t-\lambda)$ on the λ axis

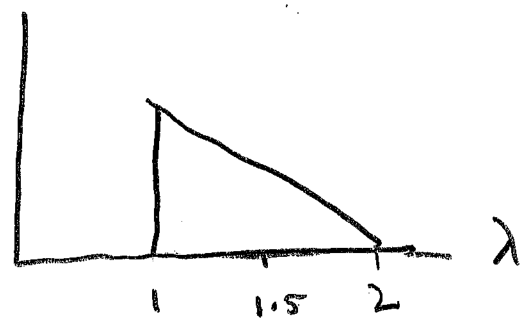
set $t = 2$ so plot $x(2-\lambda)$

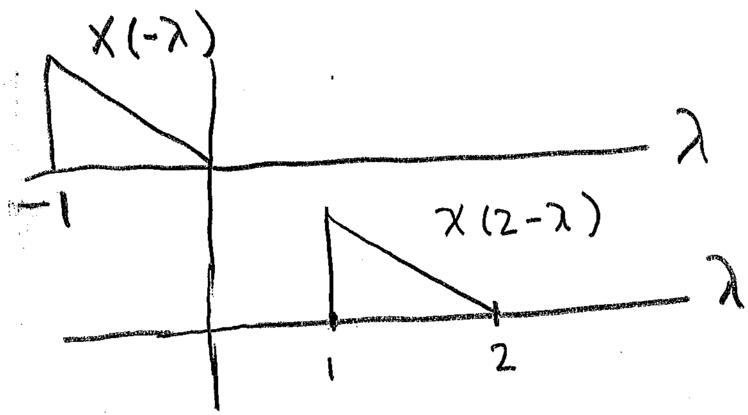
thing in parenthesis

$\lambda = 2$, $(\cdot) = (2-2) = 0$, $x = 0$

$\lambda = 1.5$, $(\cdot) = (2-1.5) = 0.5$, $x = 0.5$

$\lambda = 1$, $(\cdot) = (2-1) = 1$, $x = 1$





As t increases, the function $x(t-\lambda)$ shifts right

So given $x(t)$ we want to place $x(t-\lambda)$ on the λ axis.

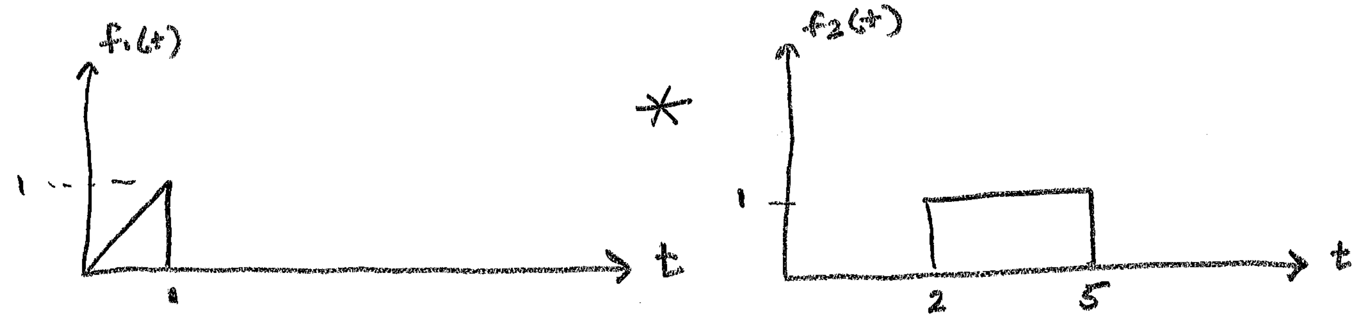
1) Draw the λ axis.

2) Mirror image x

3) shift the mirrored image right 't' units

Graphical Convolution

Ex - Pr 15.44 b



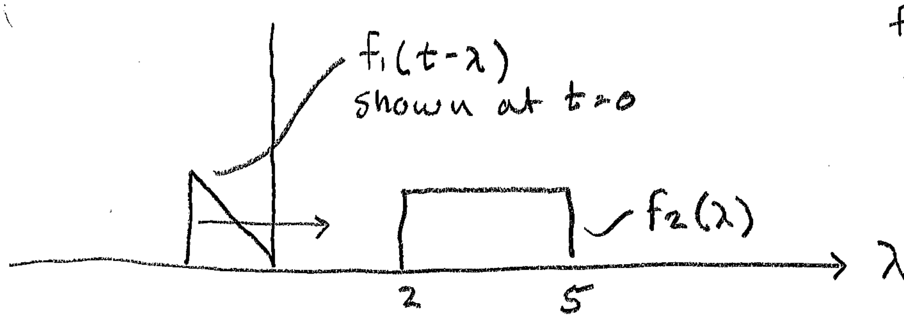
Convolution is communitive $f_1 * f_2 = f_2 * f_1$. It's easier for me if I slide my triangle.

Convolution integral

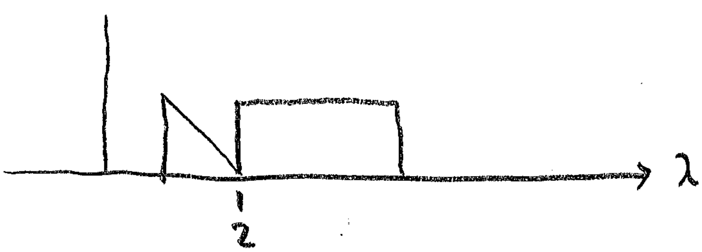
$$y(t) = \int_{-\infty}^{\infty} f_2(\lambda)\, f_1(t-\lambda)\, d\lambda$$

Visualize the integral

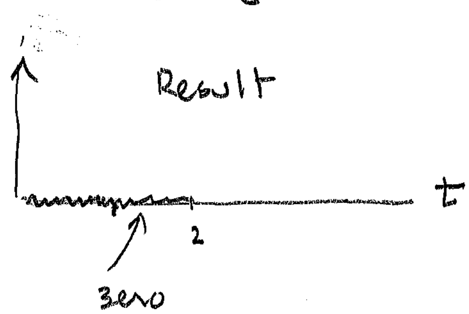
As t increases f_1 slides right. Result of integral is the area shared by both functions

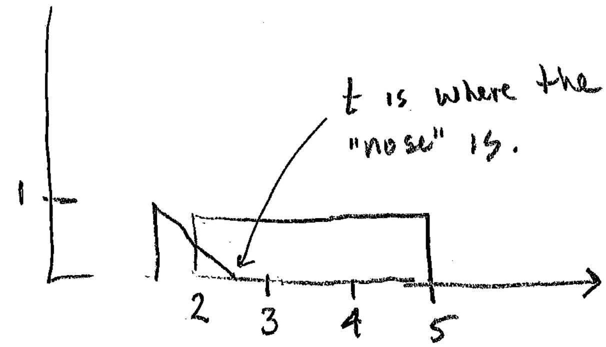


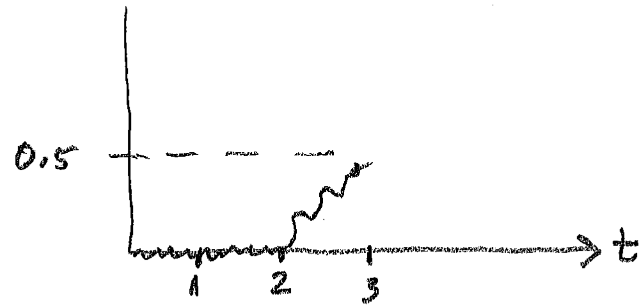
full integration is performed at every value of t.



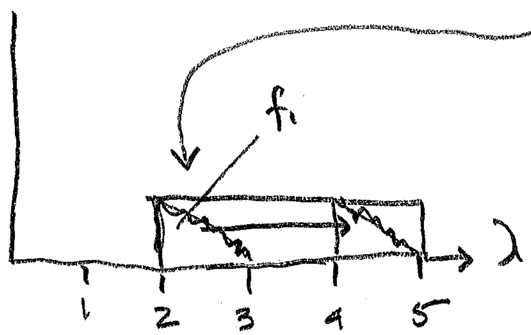
at $t < 2$, there is no overlap so $y(t) = 0$

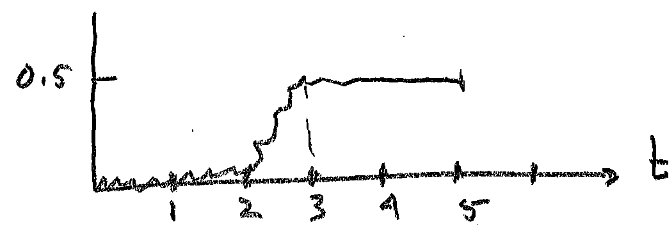




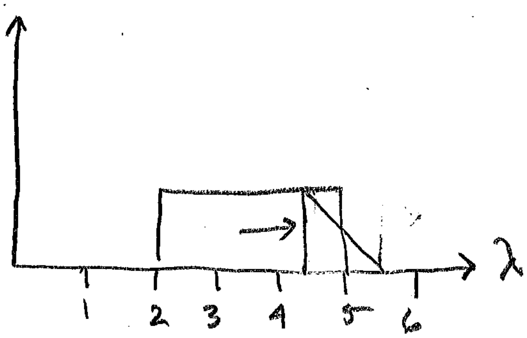


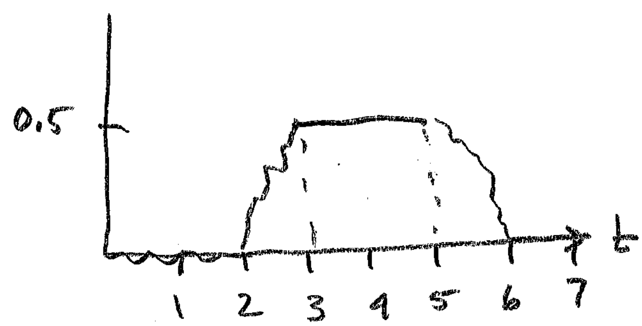
Between $t=2$ and $t=3$, the functions begin to overlap. Area increases as $f_1(\lambda)$ moves right.
At $t=3$ they overlap completely. $\frac{1}{2}bh = 0.5$





Between $t=3$ and $t=5$, the area will stay constant at 0.5

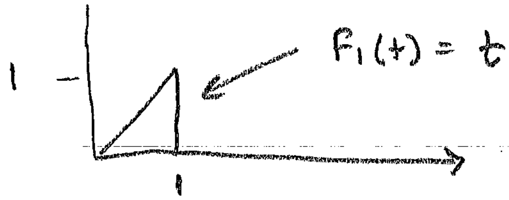




Between $t=5$ and $t=6$, the area will decrease. At $t=6$ it is zero.

Now the math...

Below $t = 2$ $\quad f(t) = 0$



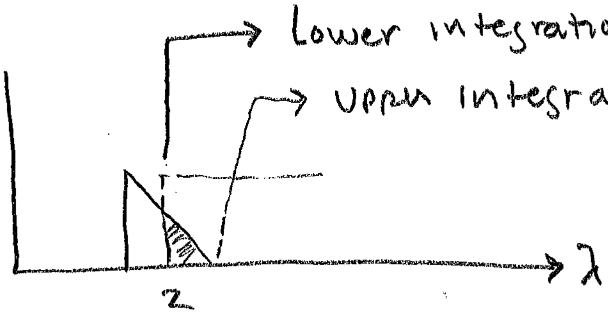
time shifted and reversed:

$f_1(t, \lambda) = t - \lambda$

$2 < t < 3$

Lower integration limit is 2

Upper integration limit is t



$$\Big|_2^t \lambda t - \frac{\lambda^2}{2} = t^2 - \frac{t^2}{2} - 2t + \frac{2^2}{2}$$

$$= \frac{t^2}{2} - 2t + 2$$

$$f(t) = \int_2^t t - \lambda \, d\lambda = \frac{t^2}{2} - 2t + 2, \quad 2 < t < 3$$

<u>Sanity check</u> $\quad 2 - 4 + 2 \qquad \frac{9}{2} - 6 + 2 = \frac{9 - 12 + 4}{2} = 1/2$

$f(2) = 0$, good, $\quad f(3) = 1/2$, good

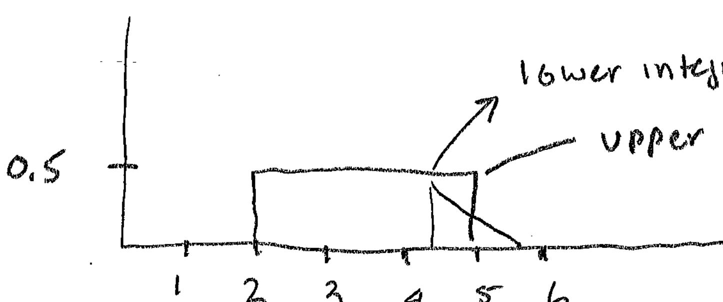
$\to = 1/2$ base $\cdot$ height

Next region: $3 < t < 5$. In this region, $f(t) = 1/2$, $3 < t < 5$

Next region: $5 < t < 6$

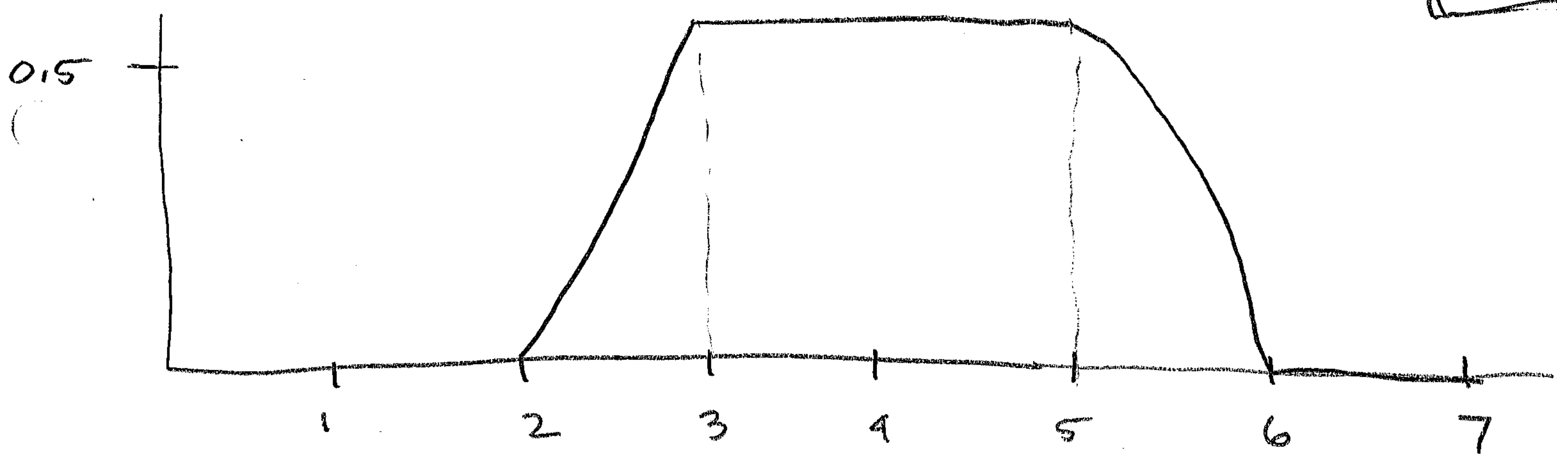
lower integration limit is $t-1$

upper integration limit is 5



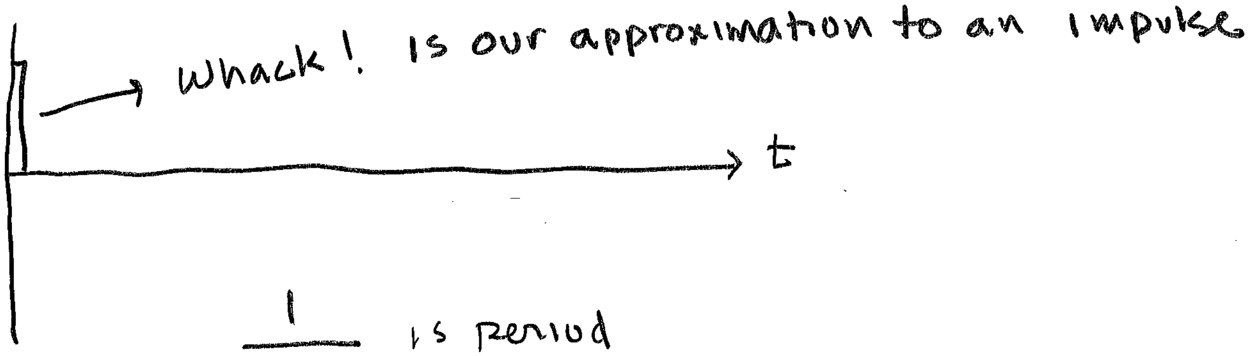
$$f(t) = \int_{t-1}^5 t - \lambda \, d\lambda = -\frac{t^2}{2} + 5t - 12, \quad 5 < t < 6$$

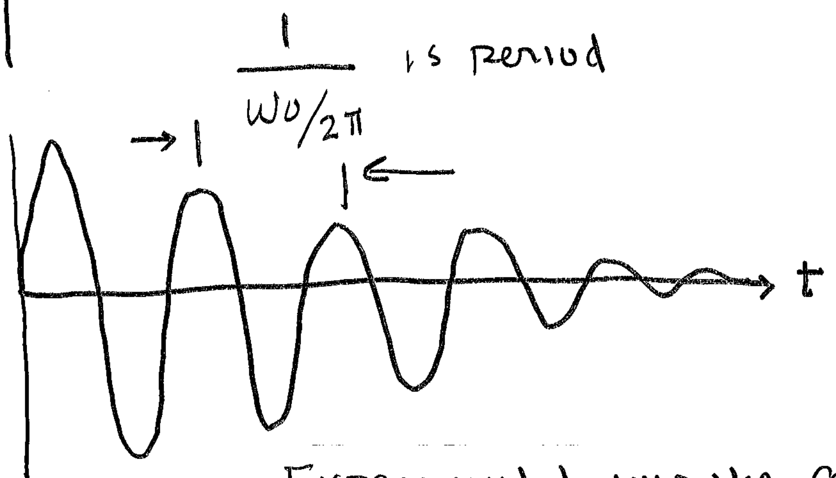
Sanity check, $f(5) = 0.5$, $\quad f(6) = 0$



See Excel Plot   Note it agrees with our estimate

PRACTICAL CONVOLUTION

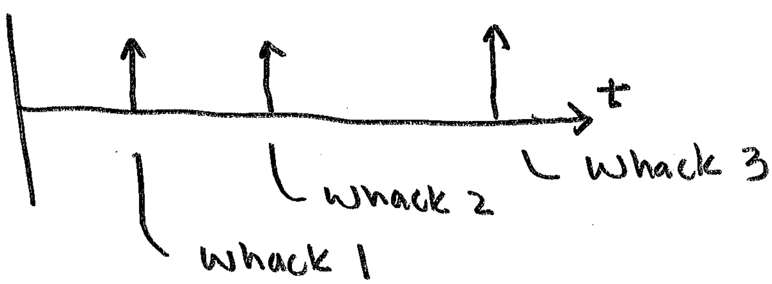




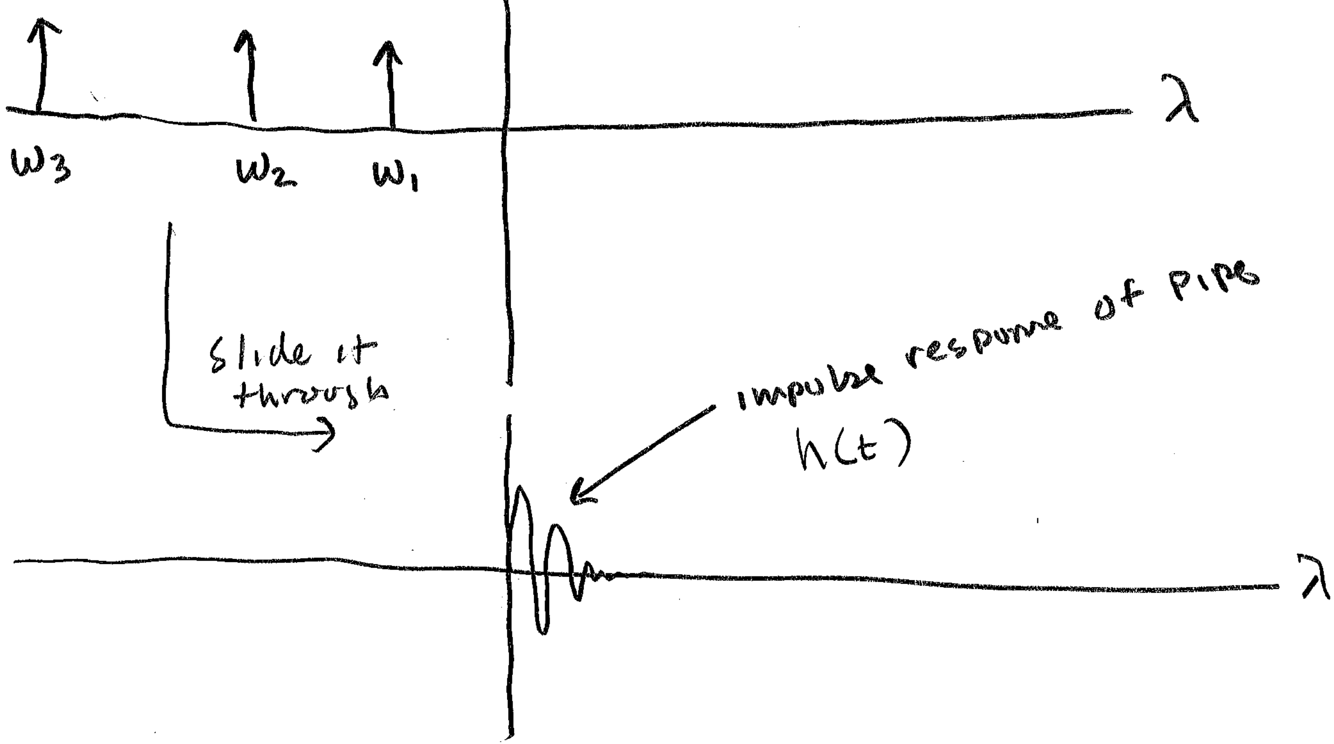
Experimental impulse response of pipe

Now let's take something easy and make it hard

Strike the pipe a few times

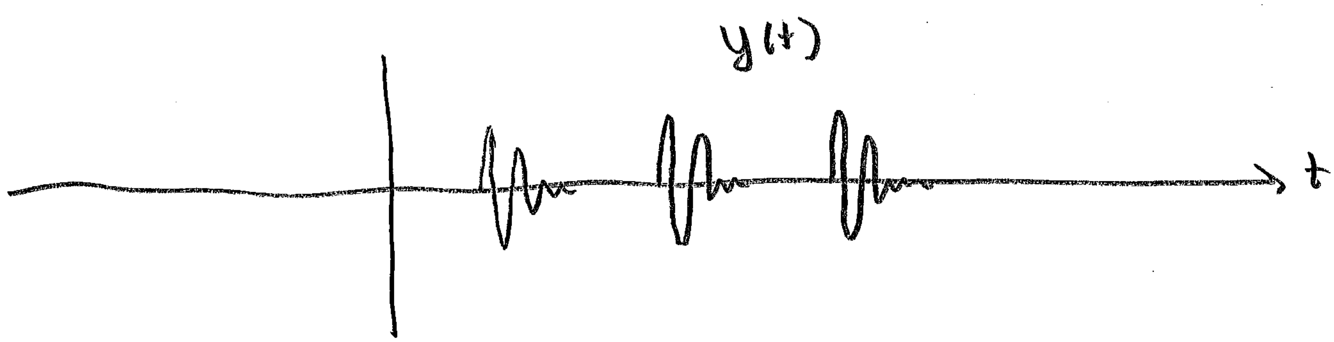


time reverse $x(t)$ and place on λ axis

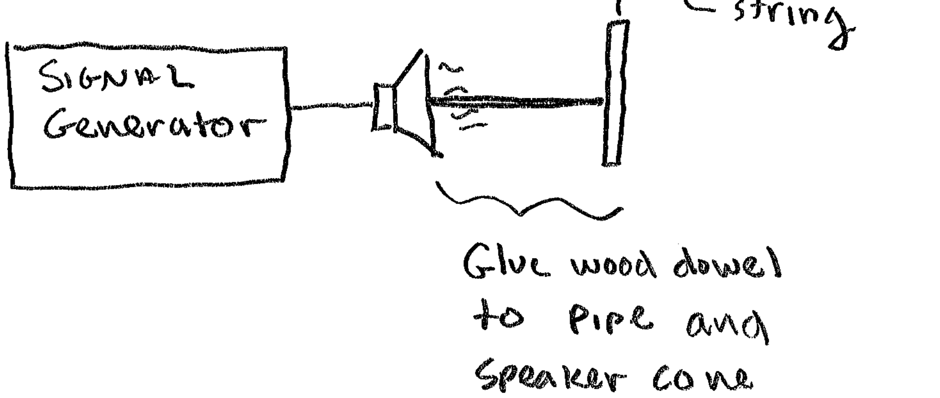


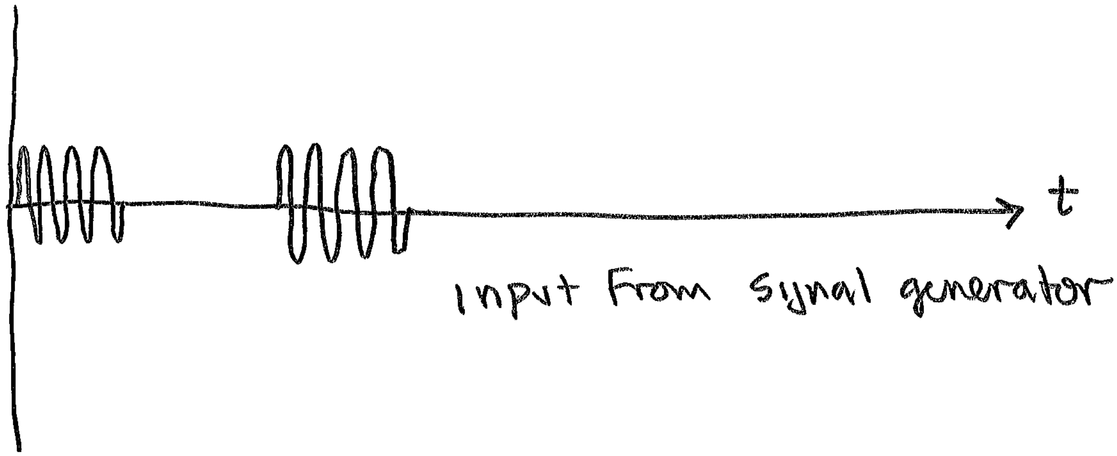
To get time domain function we slide input right and integrate as we go

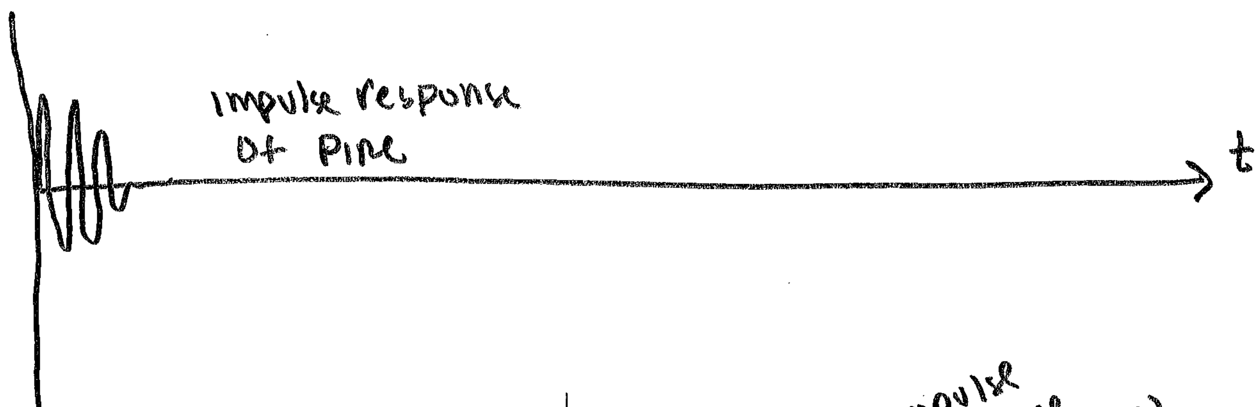
$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

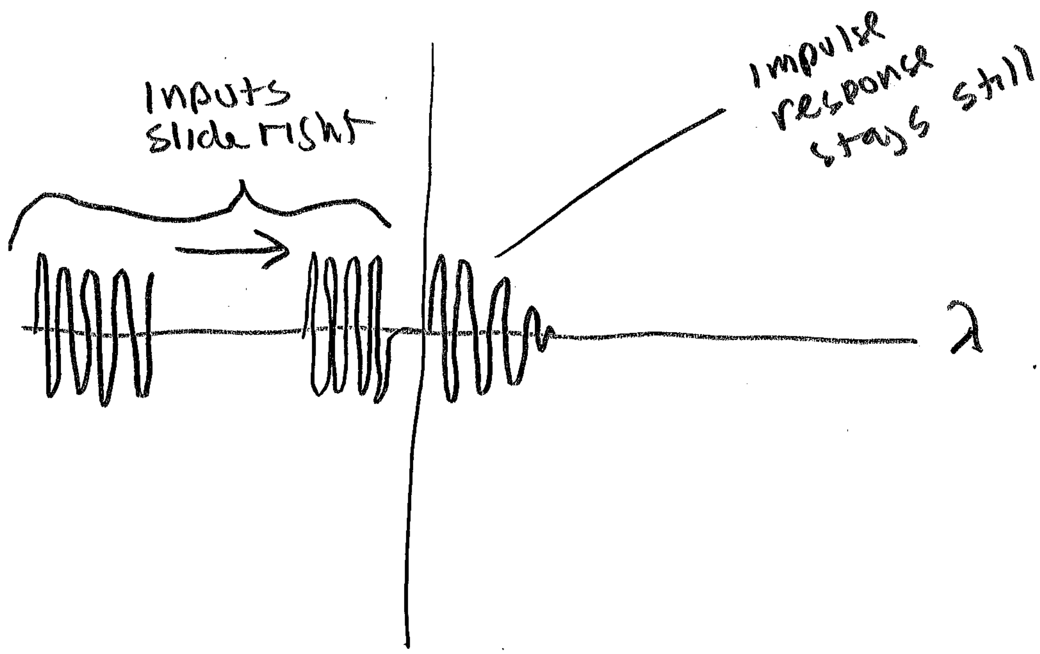


EXCITATION WITH SINEWAVE BURSTS

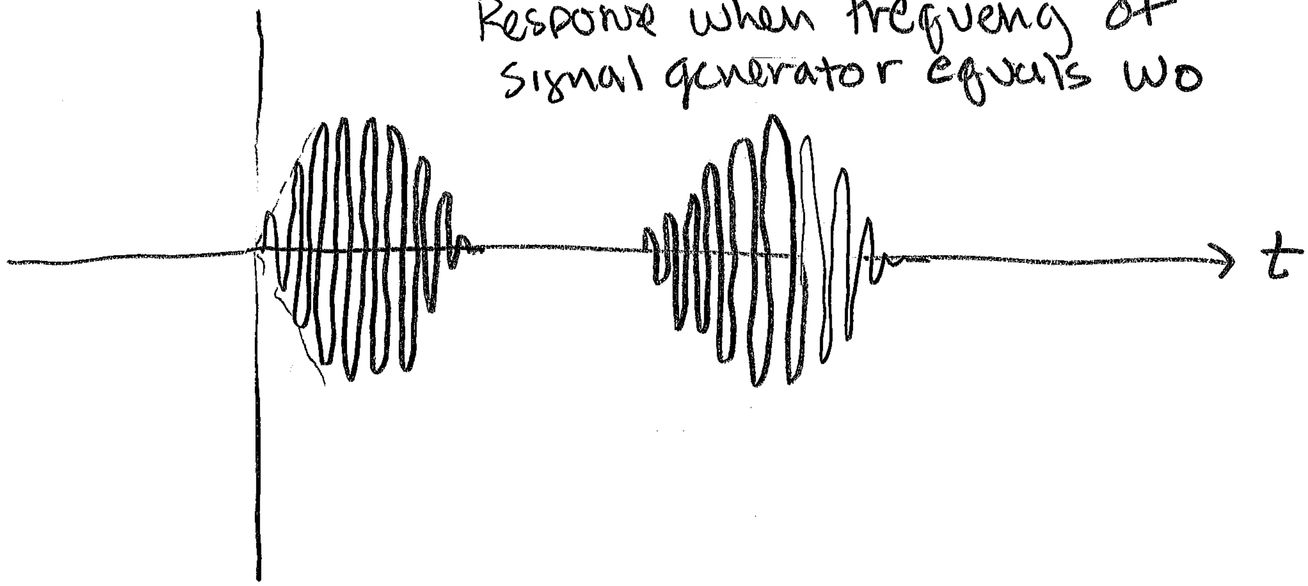




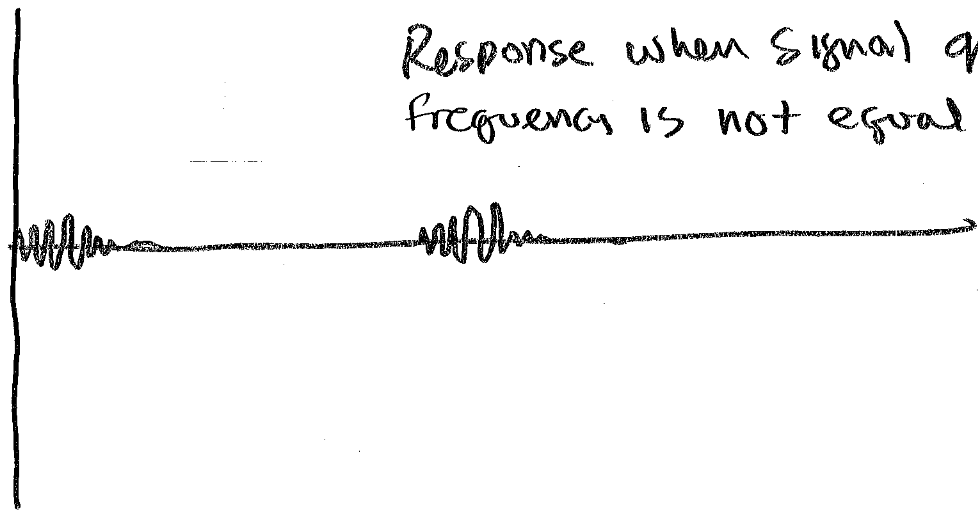




Response when frequency of signal generator equals w_0



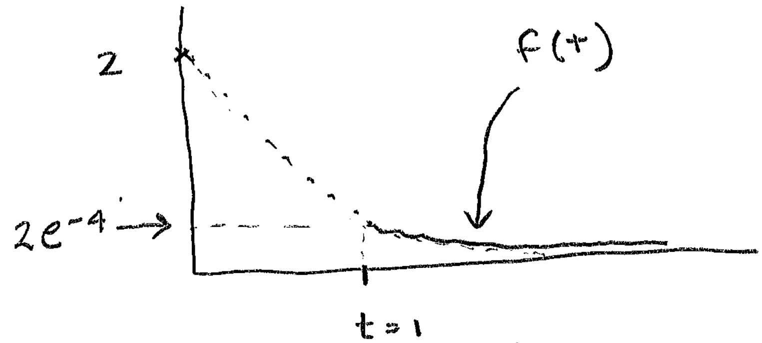
Response when signal generator frequency is not equal to w_0



So if we excite the system at its resonant frequency we get a big response

PR 15.9b - Find $F(s)$

$f(t) = 2e^{-4t}\,u(t-1)$



Strategy: 1) use properties to get the simplest $f(t)$ we can.

2) Get $F(s)$ of simplest $f(t)$

3) apply properties to get final $F(s)$

$f(t) = 2e^{-4t}\,u(t-1)$

Frequency shift $\rightarrow$ $e^{-at} f(t) \Longleftrightarrow F(s+a)$

so drop the e^{-4t} and

$f(t) = 2\,u(t-1)$

Time shift $\rightarrow$ $f(t-a)\,u(t-a) \Longleftrightarrow e^{-as} F(s)$

so drop the delay and now

$f(t) = 2\,u(t)$ $\leftarrow$ simplest $f(t)$

$\downarrow$

$F(s) = 2 \times \frac{1}{s} = \frac{2}{s}$

apply time shift of 1 second

$F(s) = \frac{2}{s} \cdot \underbrace{e^{-s}}_{\text{from time shift}}$

apply frequency shift

$F(s) = \frac{2\;e^{-(s+4)}}{(s+4)}$ from frequency shift

$$\boxed{F(s) = \frac{2e^{-(s+4)}}{s+4}}$$