LECTURE 22 - INVERSE LAPLACE TRANSFORM

Laplace xform

$$\mathcal{L}(f(t)) = F(s) = \int_{0^-}^{\infty} f(t) e^{-st}\, dt \qquad \text{EQ 15.1}$$

Inverse Laplace transform

$$\mathcal{L}^{-1}(F(s)) = f(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s) e^{st}\, ds \qquad \cancel{\text{Eq 15.5}}$$

we don't use this

Instead, we use the Laplace transform table and properties in Tables 15.1 and 15.2 pg. 687

Basic Strategy (Pg 690)

1) Decompose F(s) into a sum of simple terms using partial fraction expansion
2) Find the inverse of each term by matching entries in table 2.

Inverse Laplace transform strategy

$$F_{big}(s) = \underbrace{\frac{n_0 s^0 + n_1 s^1 + n_2 s_2 + \dots n_k s^k}{d_0 s^0 + d_1 s^1 + d_2 s_2 + \dots d_k s^k}}$$

$\downarrow$

Partial fraction Technique

$\downarrow$

$$\mathcal{L}^{-1}(F_{big}(s)) = \mathcal{L}^{-1}(F_{l1}(s)) + \mathcal{L}^{-1}(F_{l2}(s)) + \dots$$ little terms

$\downarrow \quad \downarrow$

TABLE Lookup

$\downarrow \quad \downarrow$

$$f_{big}(t) = f_{l_1}(t) + f_{l_2}(t) + f_{l_3}(t) \dots$$

$F_{big}(s)$ can contain three kinds of terms

$$F(s) = \frac{n(s)}{\underbrace{(s+p_1)(s+p_2)}_{\text{Simple Poles}}\underbrace{(s+p_3)^n}_{\text{Repeated Poles}}\underbrace{(s^2+as+b)}_{\text{Complex pole}}}$$

Simple Poles Sec 15.4.1

Repeated Poles Sec 15.4.2

Complex pole Sec 15.4.3

EQ 15.49 Guarantees

Simple Poles

$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)\ldots(s+p_n)} = \underbrace{\frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \ldots \frac{K_n}{s+p_n}}_{\text{Simple inverse transforms}}$$

residue method

$$(s+p_1)F(s) = K_1 + \frac{(s+p_1)K_2}{s+p_2} + \frac{(s+p_1)K_3}{(s+p_3)}$$

↳ cancels with term in denominator

Set $s = -p_1$

These terms go to zero leaving K_1

So in general $\quad K_i = (s+p_i)F(s)\Big|_{s=-p_i}$

and $\quad f(t) = \left[K_1 e^{-p_1 t} + K_2 e^{-p_2 t} \ldots\right] u(t)$

Example - $\quad F(s) = \dfrac{2s+4}{(s+3)(s+7)}$

$$\frac{2s+4}{(s+3)(s+7)} = \frac{K_1}{s+3} + \frac{K_2}{s+7}$$

$$K_1 = (s+3)F(s)\Big|_{\substack{s=-p_1 \\ (=-3)}} = \cancel{(s+3)}\,\frac{2s+4}{\cancel{(s+3)}(s+7)}\Bigg|_{s=-3}$$

$$= \frac{2s+4}{s+7}\Bigg|_{s=-3} = \frac{-2}{4} = -\frac{1}{2}$$

Example - Cont

$$K_2 = (s+7)\,F(s)\Big|_{\substack{s=-P_2\\(=-7)}} = \frac{2s+4}{s+3}\Big|_{s=-7} = \frac{-10}{-4} = \frac{5}{2}$$

so $$F(s) = \frac{2s+4}{(s+3)(s+7)} = -\frac{1}{2} \times \frac{1}{s+3} + \frac{5}{2} \times \frac{1}{s+7}$$

and $$f(t) = \left[K_1 e^{-P_1 t} + K_2 e^{-P_2 t}\right] u(t)$$ ← inverse Laplace

$$f(t) = \left[-\frac{1}{2} e^{-3t} + \frac{5}{2} e^{-7t}\right] u(t)$$

Let's check the answer

initial value: $f(t=0) = -\frac{1}{2} + \frac{5}{2} = 2$

from initial value theorem,

$$\underbrace{f(0) = \lim_{s\to\infty} s\,F(s)}_{\text{EQ 15.42}} = \lim_{s\to\infty}\left[\frac{s(2s+4)}{(s+3)(s+7)}\right]$$

$$\lim_{s\to\infty}\left[\frac{2s^2+4s}{s^2+10s+21}\right] = 2$$ IT checks

Now use final value theorem

$$f(\infty) = -\frac{1}{2}e^{-\infty} + \frac{5}{2}e^{-\infty} = 0$$

Example Cont

$$F(\infty) = \lim_{s\to 0}\left[s\,F(s)\right] = \lim_{s\to 0}\left[\frac{2s^2+4s)}{s^2+10s+21}\right] = 0$$

IT Checks!

Repeated Poles (sec 15.4.2)

IF $F(s)$ has a repeated pole, $F(s) = \dfrac{n(s)}{(s+p)^n}$

then $F(s)$ can be represented as (GUARANTEE)

$$F(s) = \frac{K_n}{(s+p)^n} + \frac{K_{n-1}}{(s+p)^{n-1}} \cdots \frac{K_1}{(s+p)} + F_1(s) \qquad \text{EQ } 15.54$$

remaining part of $F(s)$

Example

$$F(s) = \frac{8s+3}{(s+4)^2\cdot(s+1)}$$

$$= \frac{K_2}{(s+4)^2} + \frac{K_1}{s+4} + \frac{K_3}{s+1}$$

using EQ 15.55

$$K_2 = (s+p)^2\,F(s)\Big|_{s=-p}$$

$$= \frac{\cancel{(s+4)^2} \times (8s+3)}{\cancel{(s+4)^2}\,(s+1)}\Bigg|_{s=-4} = \frac{-29}{-3} = \frac{29}{3}$$

Use EQ 15.56 to get K_1

$$K_1 = \frac{d}{ds}\left[(s+p)^n F(s)\right]\Big|_{s=-p}$$

$$K_1 = \frac{d}{ds}\left[\frac{8s+3}{s+1}\right]\Big|_{s=-4}$$

I only remember the product rule!

$$K_1 = \frac{d}{ds}\left[(8s+3)(s+1)^{-1}\right]\Big|_{s=-4}$$

$$K_1 = \left[\frac{8}{s+1} - \frac{(8s+3)}{(s+1)^2}\right]\Big|_{s=-4} = \frac{8}{-3} - \frac{-29}{9}$$

$$= \frac{-24+29}{9} = \frac{5}{9} = K_1$$

Compute K_3 - It is a simple pole. Use EQ 15.52

$$K_3 = (s+p)\,F(s)\Big|_{s=p}$$

$$K_3 = (s+1)\,F(s)\Big|_{s=-1} = \frac{8s+3}{(s+4)^2}\Big|_{s=-1} = \frac{-5}{9} = K_3$$

$$\text{So } F(s) = \frac{8s+3}{(s+4)^2(s+1)} = \frac{29}{3}\times\frac{1}{(s+4)^2} + \frac{5}{9}\times\frac{1}{(s+4)} - \frac{5}{9}\times\frac{1}{s+1}$$

EQ 15.54

Use Table 15.2 to get f(t)

$$F(s) = \underbrace{\frac{29}{3} \times \frac{1}{(s+4)^2}}_{} + \underbrace{\frac{5}{9} \times \frac{1}{s+4}}_{} - \underbrace{\frac{5}{9} \times \frac{1}{s+1}}_{}$$

$$f(t) = \frac{29}{3} \times t e^{-4t} + \frac{5}{9} e^{-4t} - \frac{5}{9} e^{-t}$$

Check It

$$t = 0, \quad f(t) = 0 + 5/9 - 5/9 = 0$$

$$f(0) = \lim_{s \to \infty} \left[s\, F(s) \right] = \lim_{s \to \infty} \left[\frac{s(8s+3)}{(s+4)^2 (s+1)} \right] = 0$$

degree of denominator is greater

Method of Algebra

We can solve any partial fraction with this "Brute Force" method. (including complex poles which we haven't covered yet.)

~ Sometimes it's easier.

~ Sometimes it isn't.

Return to our first example using method of algebra

$$F(s) = \frac{2s+4}{(s+3)(s+7)} = \underbrace{\frac{K_1}{s+3} + \frac{K_2}{s+7}}$$

EQ 15.49 Guarantees this will hold

So just solve it

$$\frac{2s+4}{(s+3)(s+7)} = \frac{K_1(s+7) + K_2(s+3)}{(s+3)(s+7)} = \frac{K_1 s + 7K_1 + K_2 s + 3K_2}{(s+3)(s+7)}$$

EQuate the numerators

$$2s+4 = K_1 s + 7K_1 + K_2 s + 3K_2$$
$$= s(K_1 + K_2) + 7K_1 + 3K_2$$

So $2 = K_1 + K_2$ ← equate s terms

$4 = 7K_1 + 3K_2$

$$K_1 + K_2 = 2$$
$$7K_1 + 3K_2 = 4$$

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 7 & 3 \end{bmatrix}^{-1} \times \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 2.5 \end{bmatrix}$$

which is what we got before

Return to our second example using method of algebra

$$F(s) = \frac{8s+3}{(s+4)^2(s+1)} = \underbrace{\frac{K_2}{(s+4)^2} + \frac{K_1}{(s+4)} + \frac{K_3}{s+1}}_{\text{EQ 15.59 Guarantees this}}$$

$$\frac{8s+3}{(s+4)^2(s+1)} = \frac{K_2(s+1) + K_1(s+4)(s+1) + K_3(s+4)^2}{(s+4)^2 \cdot (s+1)}$$

$$= K_2 s + K_2 + K_1(s^2+5s+4) + K_3(s^2+8s+16)$$

$$= K_2 s + K_2 + K_1 s^2 + 5K_1 s + 4K_1 + K_3 s^2 + 8K_3 s + 16K_3$$

$$= s^2\underbrace{(K_1+K_3)}_{=0} + s\underbrace{(K_2+5K_1+8K_3)}_{=8} + \underbrace{K_2+4K_1+16K_3}_{=3}$$

EQUATE

$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 1 & 8 \\ 4 & 1 & 16 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix}$$

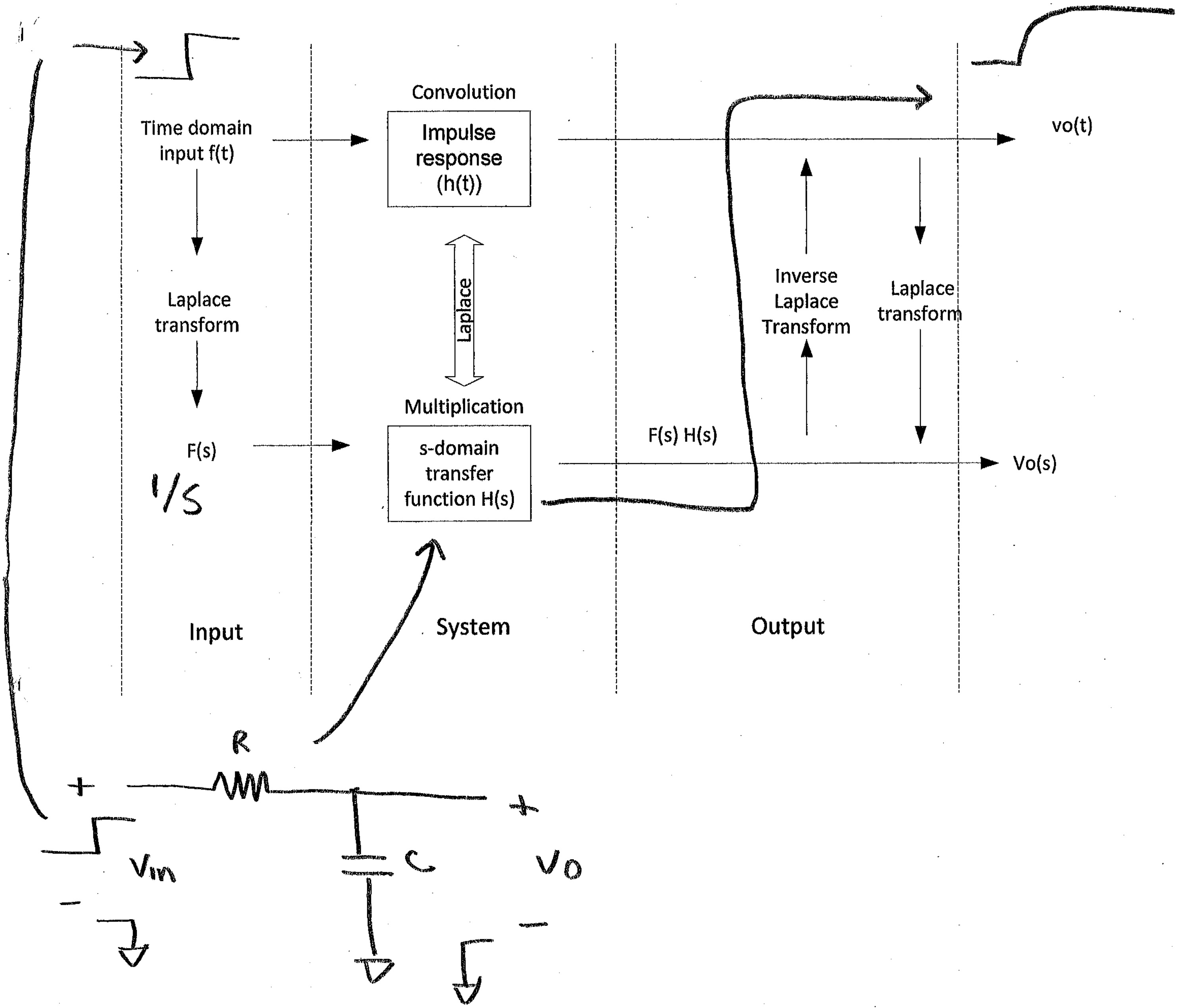
$K_1 = 5/9$

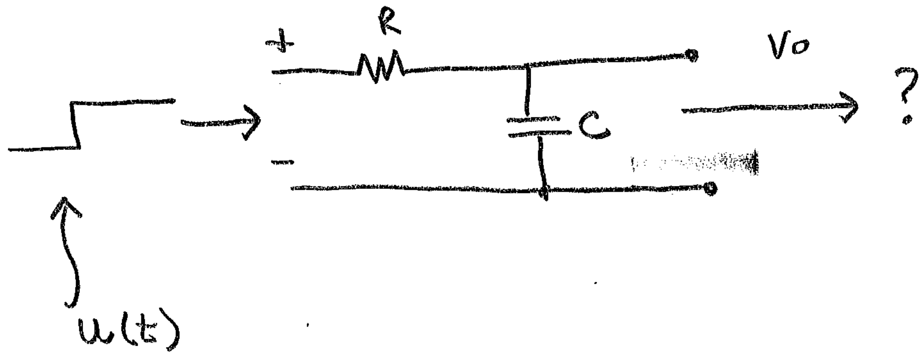
$K_2 = 29/3$

$K_3 = -5/9$

Circuit Example

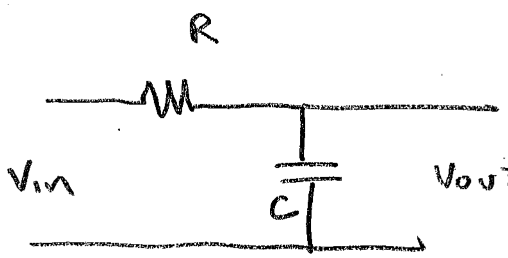
Find response of RC Circuit to a unit step





1) Get Laplace transform of input $\rightarrow$ $\mathcal{L}\{u(t)\} = \frac{1}{s}$
2) Compute transfer function for circuit. $H(s) = \frac{V_o(s)}{V_{in}(s)}$
3) MULTIPLY Laplace transform of input by the transfer function. Gives Laplace transform of output.
4) Invert Laplace transform of output to get time domain representation of output.

Compute transfer function of circuit.



$$H(s) = \frac{V_o}{V_{in}} = \frac{1/sC}{1/sC + R} = \frac{1}{1 + sCR}$$

$$H(s) = \frac{1}{RC} \times \frac{1}{s + 1/RC}$$

$$\mathcal{L}\{\text{OUTPUT}\} = \mathcal{L}\{\text{INPUT}\} \times H(s)$$

$$\mathcal{L}\{V_o\} = \frac{1}{s} \times \frac{1}{RC} \times \frac{1}{s + 1/RC} = \frac{1}{RC} \times \frac{1}{s} \times \frac{1}{s + 1/RC}$$

Invert Laplace transform of OUTPUT to get time representation of output

Use Time integration property

$$\int_0^t f(x)\,dx \Longleftrightarrow \frac{1}{s}F(s) \quad \text{or} \quad \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t \mathcal{L}^{-1}\{F(s)\}\,dt$$

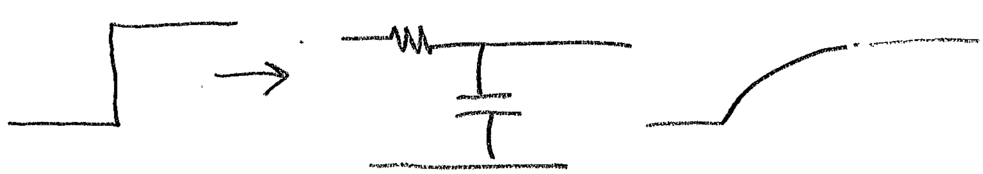
$$OUT(s) = \frac{1}{RC} \times \frac{1}{s} \times \frac{1}{s+1/RC}, \quad \text{so} \quad F(s) = \frac{1}{s+1/RC}$$

$$F(s) = \frac{1}{s+1/RC} \quad \text{so} \quad f(t) = e^{-t/RC}$$ — From table

$$\text{and} \quad \frac{1}{RC}\int_0^t e^{-t/RC}\,dt = \frac{1}{RC}\Big|_0^t -RC \times e^{-t/RC}$$

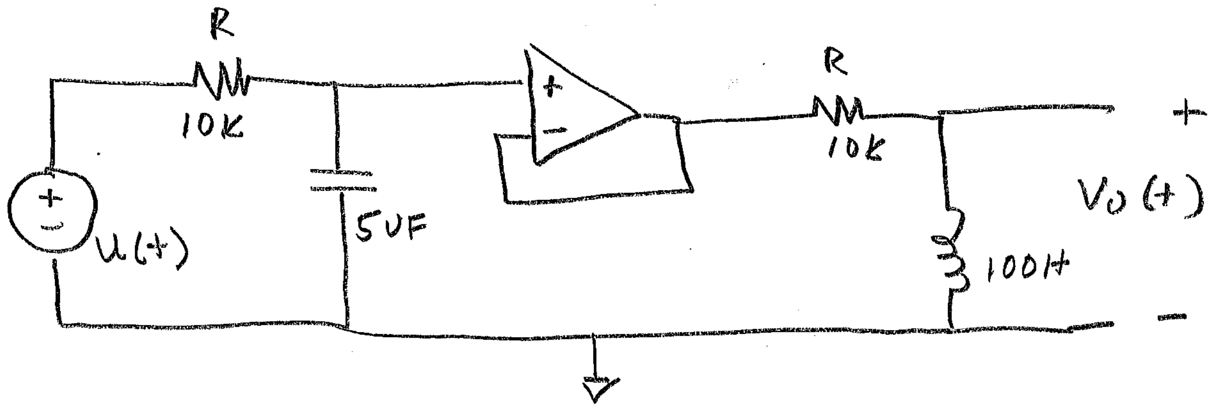
$$= \Big|_0^t -e^{-t/RC} = -\left(e^{-t/RC}-1\right) = 1-e^{-t/RC}$$

so



capacitor voltage can't change quickly,
No capacitor current at DC.

EXAMPLE – Find Vo(t)



what is $V_0(0^+)$? what is $V_0(\infty)$

Laplace xform of input, $u(t) \longrightarrow 1/s$

Transfer function for Circuit

First stage

$$H_1(s) = \frac{1/sC}{R + 1/sC} = \frac{1}{1+sCR} = \frac{1}{RC} \times \frac{1}{s + 1/RC}$$

second stage

$$H_2(s) = \frac{sL}{sL+R} = \frac{s}{s + R/L}$$

So Cascaded transfer function is

$$H(s) = H_1(s) \times H_2(s) = \frac{1}{RC} \frac{1}{(s+1/RC)} \times \frac{s}{(s + R/L)}$$

$$V_o(s) = V_{in}(s) \times H(s) = \frac{1}{s} \times H(s)$$

$$V_o(s) = \frac{1}{RC} \times \frac{1}{(s+1/RC)} \times \frac{1}{(s+R/L)}$$

$$V_o(t) = \mathcal{L}^{-1}\{V_o(s)\}$$

guarantee

$$V_o(s) = \frac{1}{RC} \times \frac{1}{(s+1/RC)} \times \frac{1}{(s+R/L)} = \frac{K_1}{(s+1/RC)} + \frac{K_2}{(s+R/L)}$$

Use Method of Algebra

$$\frac{K_1(s+R/L) + K_2(s+1/RC)}{(s+1/RC)(s+R/L)} = \frac{1}{RC}\,\frac{1}{(s+1/RC)(s+R/L)}$$

or $K_1(s+R/L) + K_2(s+1/RC) = \frac{1}{RC}$

$$K_1 s + K_1 R/L + K_2 s + \frac{K_2}{RC} = \frac{1}{RC}$$

$$\left.\begin{aligned} s(K_1 + K_2) &= 0 \\ K_1\frac{R}{L} + \frac{K_2}{RC} &= \frac{1}{RC} \end{aligned}\right\} \text{Equate coefficients}$$

$$\begin{bmatrix} 1 & 1 \\ R/L & 1/RC \end{bmatrix}^{-1} \times \begin{bmatrix} 0 \\ 1/RC \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

for $R = 10K$, $C = 50uF$, $L = 100H$,

$K_1 = 0.25$, $K_2 = -0.25$

$\longrightarrow$

$$V_o(s) = \frac{0.25}{s+20} - \frac{0.25}{s+100}$$

invert to time domain

$$v_o(t) = 0.25e^{-20t} - 0.25e^{-100t}$$

You can rework this as a general RLC problem from Chapter 8

Recommend you check your answer with initial and final value theorems.