LECTURE 21 - LAPLACE TRANSFORM

INTRODUCTION AND PROPERTIES

CHAPTER 8 - Thuds, boings, and initial conditions

CHAPTER 9, 10, 11, 13, 14 - Sinusoidal steady state

Solution with Laplace transforms covers both cases plus ...

- Input are not limited by unit steps or sinewaves.
- Differential equations are turned into algebraic equations.
- Laplace analysis allows initial conditions to be handled in a methodical way.
- Tables of Laplace transforms simplify operations.

FROM PG 677:

"The Laplace Transform is an integral transformation of a function $f(t)$ from the time domain to the complex frequency domain giving $F(s)$"

EQ 15.1, Page 677 → anatomy

Laplace operator

Frequency domain function big F

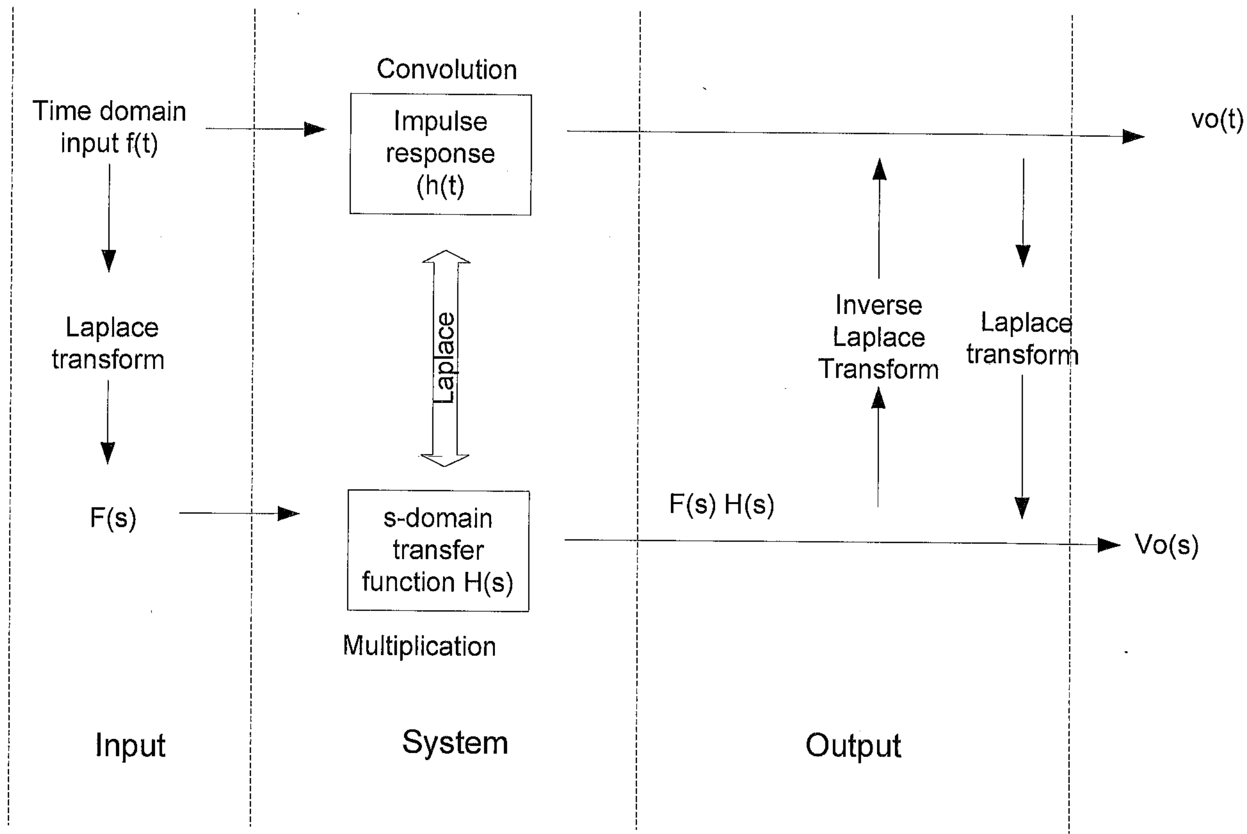
$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

Time function, little f

$s = \sigma + j\omega$

t gets integrated out leaving a function of s

time domain functions usually accompanied by u(t)



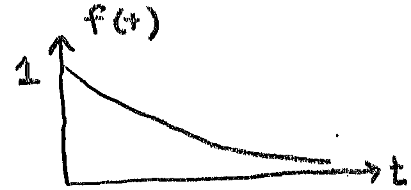
LAPLACE TRANSFORM ROADMAP

Why??

1) Start with a signal
2) Feed it through a system
3) Determine the response.

Example 15.1.b Pg 678

Find the Laplace transform $f(t) = e^{-\alpha t} u(t)$



$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-\alpha t} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s+\alpha)t}\, dt = \frac{-1}{s+\alpha} \Big|_0^\infty e^{-(s+\alpha)t}$$

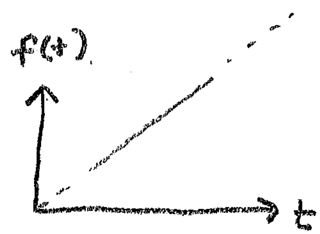
$$= \frac{-1}{s+\alpha} [0 - 1] = \frac{1}{s+\alpha}$$

Note this result in table 15.2

Practice Problem 15.1 pg 679

Find the Laplace transform of $f(t) = t\, u(t)$

"ramp function"



$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty t e^{-st}\, dt$$

$$\int_0^\infty t e^{-st}\, dt = \Big|_0^\infty \frac{e^{-st}}{s^2}(-st - 1) \longleftarrow$$

See pg A-18 or any table of integrals

$$= \frac{1}{s^2} [0 - (-1)] = \frac{1}{s^2}$$

Properties of the Laplace transform

Linearity – Eq 15.7

$$\mathcal{L}\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)$$

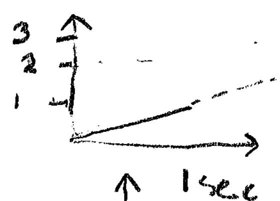
Usefulness $\longrightarrow$ given a sum of terms, you can compute Laplace transforms independently.
Lets you handle terms one at a time.

Scaling

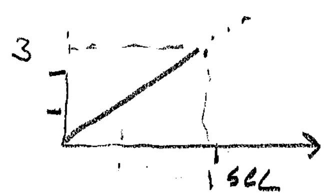
$$\mathcal{L}\left[f(at)\right] = \frac{1}{a} F\left(\frac{s}{a}\right) \qquad \text{Eq. 15.12}$$

Example, ramp function $f(t) = t$, $F(s) = \frac{1}{s^2}$ $\leftarrow$ table 15.2

$f(t) = 3t$



"normalized"



Our problem

$$\mathcal{L}[3t] = \frac{1}{3} F\left(\frac{s}{3}\right) = \frac{1}{3} \times \frac{1}{\left(\frac{s}{3}\right)^2} = 3 \times \frac{1}{s^2} = \frac{3}{s^2}$$

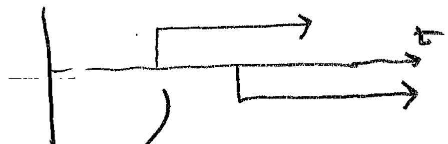
We adapted the result in the table to our problem and avoided having to integrate.

Time shift

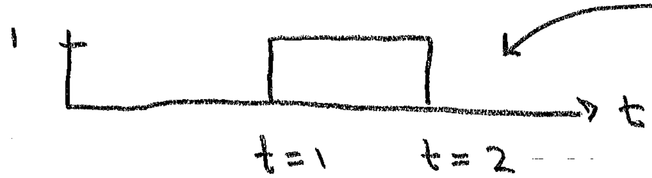
$$\mathcal{L}\left[f(t-a)\,u(t-a)\right] = e^{-as}F(s) \qquad \text{Eq 15.17}$$

Usefulness → not all functions start at zero

→ You may have multiple functions staggered in time



Example   Make a pulse



$$f(t) = \underbrace{u(t-1)}_{f_1(t)} - \underbrace{u(t-2)}_{f_2(t)}$$

Use linearity property

$$F(s) = F_1(s) + F_2(s)$$

$$F(s) = \frac{1}{s} \quad \longleftarrow \text{ from table 15.2}$$

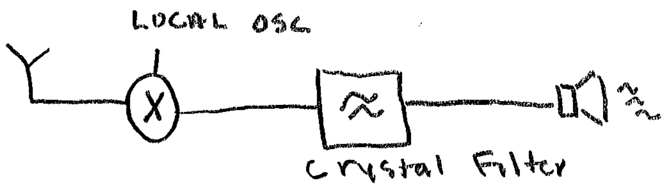
Use time shift property

$$F(s) = \underbrace{\frac{1}{s}e^{-s}}_{F_1(s)} - \underbrace{\frac{1}{s}e^{-2s}}_{F_2(s)} = \frac{1}{s}\left[e^{-s} - e^{-2s}\right]$$

Frequency Shift

$$\mathcal{L}\left[e^{-at} f(t)\, u(t)\right] = F(s+a) \quad \text{Eq } 15.19$$

Frequency? huh? $\longrightarrow$ if $a = j\omega$ we are multiplying by a sinewave.



Time Differentiation

$$\mathcal{L}\left[f'(t)\right] = s\,F(s) - f(0^-) \quad \text{Eq } 15.23$$

or $\mathcal{L}\left[f''(t)\right] = s^2 F(s) - s\,f(0^-) - f'(0^-)$

and so on

Example $\longrightarrow$

$$\underbrace{\mathcal{L}\left[\frac{di}{dt}\right] = s\,\mathcal{L}\left[i(t)\right] - i(0^-)}$$

See where this is going???

$v(t) = L\dfrac{di}{dt}$, $\mathcal{L}\{v\} = L\,\mathcal{L}\left\{\dfrac{di}{dt}\right\}$

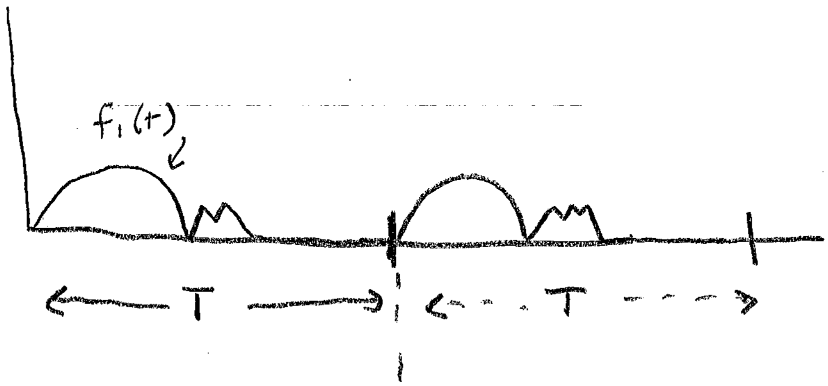
$\mathcal{L}(v) = L \cdot s\, \mathcal{L}(I) \longrightarrow \dfrac{\mathcal{L}(v)}{\mathcal{L}(i)} = sL$

Time Integration

$$\mathcal{L}\left[\int_0^t f(x)\,dx\right] = \frac{1}{s} F(s) \quad \text{Eq } 15.28$$

Time Periodicity

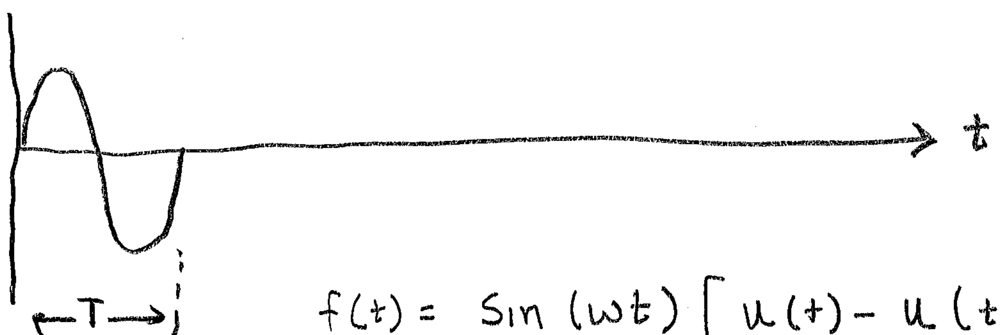
If $f(t)$ is periodic with period T then



$$F(s) = \frac{F_1(s)}{1 - e^{-sT}}$$

Eq 15.40

Example → Bridge between thuds and boings of CH 8 and steady state.



$$f(t) = \sin(\omega t)\left[u(t) - u(t-T)\right]$$

$$\mathcal{L}\{f(t)\} = \int_0^T \sin(\omega t)\, e^{-st}\, dt$$ ← (leaves function of s)

$$= \left. e^{-st} \frac{(-s \cdot \sin(\omega t) - \omega \cos(\omega t))}{s^2 + \omega^2} \right|_0^T$$ ← See pg A.19 for this integral

$$\frac{e^{-sT}\left(-s\cdot\cancelto{0}{\sin(\omega T)} - \omega\cancelto{1}{\cos(\omega T)}\right) - e^{0}\left(-s\cdot\cancelto{0}{\sin(0)} - \omega\cancelto{1}{\cos(\omega\times 0)}\right)}{s^2+\omega^2}$$

$$= \frac{\left(e^{-sT}\times -\omega\right) + \omega}{s^2+\omega^2} = \frac{\omega\left(1-e^{-sT}\right)}{s^2+\omega^2}$$

Now make it periodic via the <u>periodicity property</u>.

$$F(s) = \frac{F_1(s)}{1-e^{-sT}}$$

$$F(s) = \frac{\omega\left(1-e^{-sT}\right)}{s^2+\omega^2} \times \frac{1}{1-e^{-sT}} = \frac{\omega}{s^2+\omega^2}$$

which is in Table 15.2

Example of Laplace xform properties – PR 15-10 | 8 1/2

$$g(t) = \frac{d}{dt}\left[t e^{-t}\cos(t)\right] \qquad g(0) = 0$$

Simplify f(t)

time differentiation: $\frac{df}{dt} \Longleftrightarrow sF(s) - \cancel{f(0)}^{\,0}$

so $g(t) = t e^{-t}\cos(t)$

frequency differentiation: $t f(t) \Longleftrightarrow -\frac{d}{ds}F(s)$

so $g(t) = e^{-t}\cos(t)$

Frequency shift: $e^{-at} f(t) \Longleftrightarrow F(s+a)$

so $g(t) = \cos(t)$

$\downarrow$ LAPLACE

Simple Laplace xform

$$F(s) = \frac{s}{s^2+1}$$

Frequency shift

$$F(s) = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$$

Modify F(s)

Frequency differentiation

$$F(s) = \frac{-s(s+2)}{(s^2+2s+2)^2}$$

Time differentiation

$$F(s) = \frac{-s^2(s+2)}{(s^2+2s+2)^2}$$

Operations on frequency domain functions that provide information about time domain behavior.

Initial valve theorem

$$f(t=0) = \lim_{s \to \infty} \left[s\, F(s) \right]$$

EQ 15.42

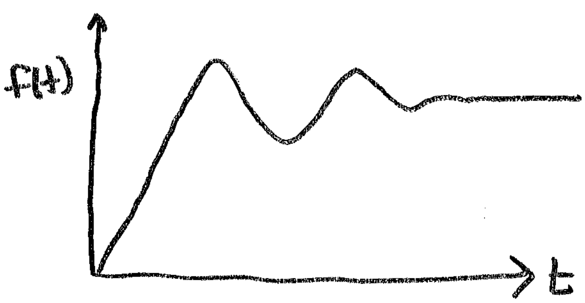
Final Valve Theorem

$$f(t=\infty) = \lim_{s \to 0} \left\{ s \cdot F(s) \right\}$$

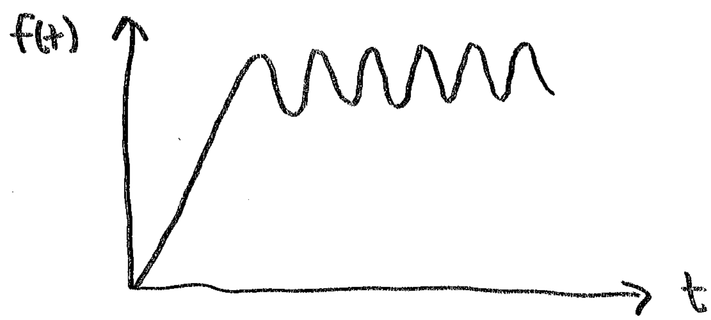
EQ 15.44

Extremely useful but we'll do more background first!

Note that for final valve to hold, the function must converge to a constant value. Pg 686 notes "all poles must be in the left hand plane"



Final Valu theorem holds



Final valve theorem does not hold.

Practice Problem 15.3 Pg 687

find Laplace transform of

$$f(t) = (\cos(2t) + e^{-4t})\,u(t)$$

does this converge? will final value theorem hold?

use linearity property

$$F(s) = \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)]$$

$$F_1(s) = \int_0^{\infty} \cos(2t)\, e^{-st}\, dt$$

use integral on pg A-19

$$F_1(s) = \left. \frac{e^{-st}}{4+s^2} \cdot \left[-s\cos(2t) + 2\sin(2t)\right] \right|_0^{\infty}$$

$$F_1(s) = \left[0 - \frac{1}{4+s^2}\left[-s\right]\right] = \frac{s}{4+s^2}$$

Could avoid integration using table entry for cosine and scaling property.

$$F_2(s) = \int_0^{\infty} e^{-4t} e^{-st}\, dt = \int_0 e^{-(s+4)t}\, dt$$

$$F_2(s) = \frac{-1}{s+4} \left. e^{-(s+4)t} \right|_0^{\infty} = \frac{-1}{s+4} \cdot [0 - 1] = \frac{1}{s+4}$$

So $F(s) = F_1(s) + F_2(s) = \dfrac{s}{4+s^2} + \dfrac{1}{s+4}$

$$= \frac{s(s+4) + 4 + s^2}{(4+s^2)(s+4)} = \frac{s^2 + 4s + 4 + s^2}{(4+s^2)(s+4)}$$

$$\boxed{F(s) = \frac{2s^2 + 4s + 4}{(4+s^2)(s+4)}}$$

Say you are given this Laplace transform
What is the value of $f(t)$ at $t=0$?
What is the value of $f(t)$ at $t=\infty$?

initial value theorem $\quad f(0) = \lim\limits_{s \to \infty} [s F(s)]$

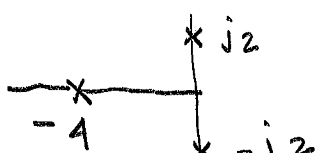
$$f(0) = \lim_{s \to \infty} \left[\frac{s(2s^2+4s+4)}{(4+s^2)(s+4)} \right]$$

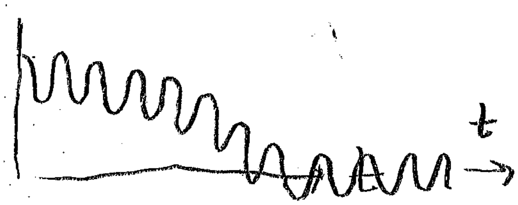
$$= \lim_{s \to \infty} \left[\frac{2s^3 + 4s^2 + 4s}{s^3 + 4s^2 + 4s + 16} \right] = 2$$

really ??

$$f(0) = \cos(0) + e^{-4 \times 0} = 2$$

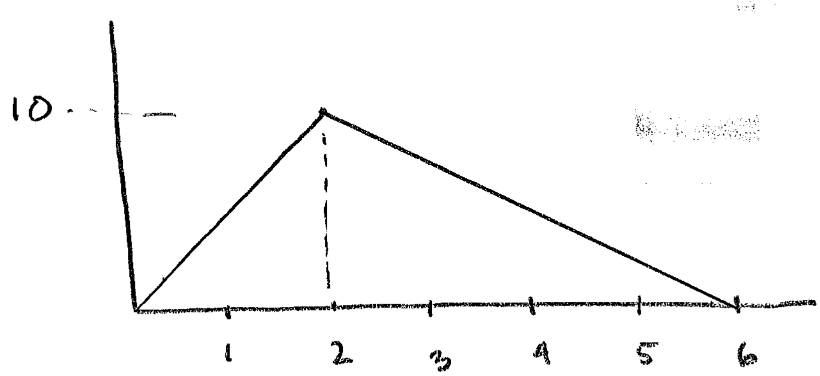
Final value $\rightarrow$ can't do because poles of $F(s)$ are not all in left half plane.





Problem 15.14

Find the Laplace transform



First segment $y = \frac{10}{2} x = 5x$

second segment

Points

	x	Y
Point 1	2	10
Point 2	6	0

"two point form" wikipedia

$$Y - Y_1 = \frac{Y_2 - Y_1}{X_2 - X_1}(X - X_1)$$

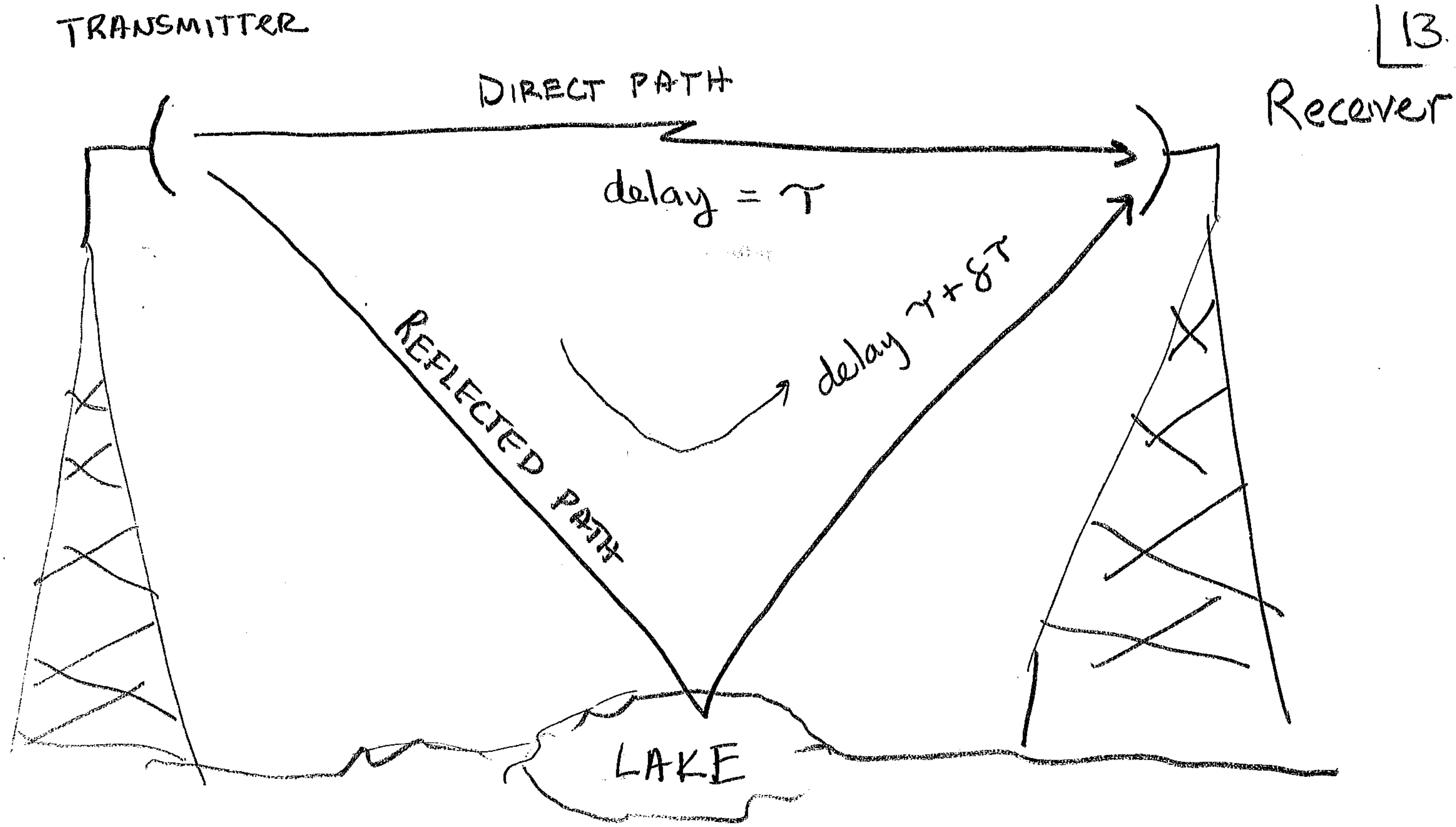
$$Y - 10 = \frac{0 - 10}{6 - 2}(X - 2)$$

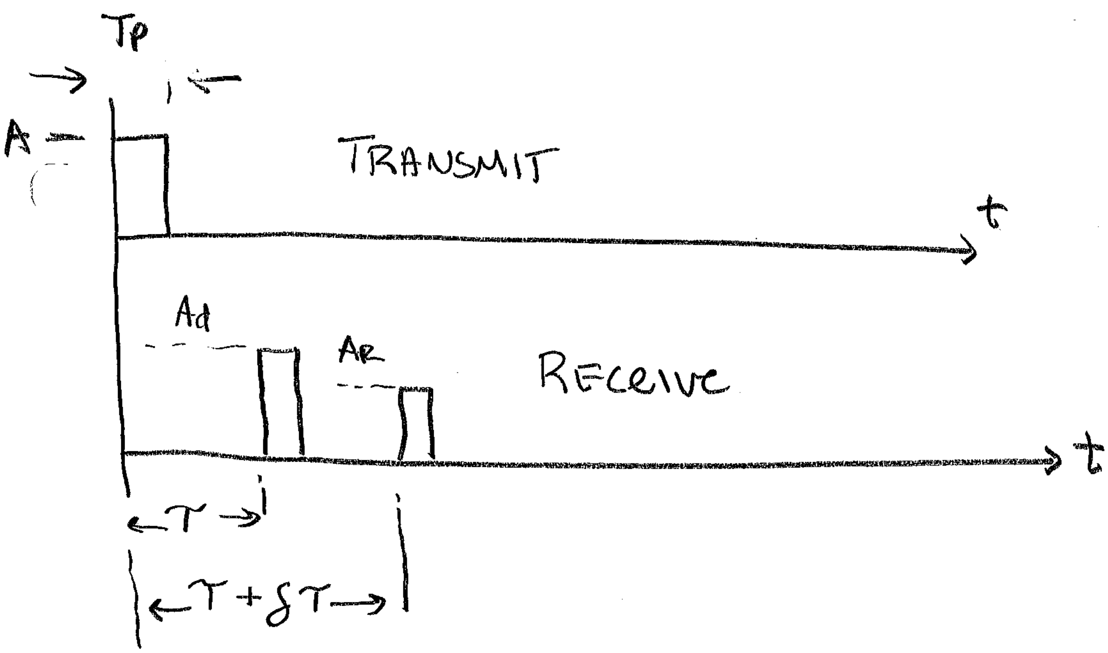
$$Y = -\frac{5}{2}x + 15$$

$$\mathcal{L}(f(t)) = \int_0^2 5t\, e^{-st}\, dt + \int_2^6 (-2.5t + 15) e^{-st}\, dt$$

$$= 5\int_0^2 t e^{-st} dt - 2.5\int_2^6 t e^{-st} dt + 15\int_2^6 e^{-st} dt$$

These integrals are on pg A-18





d is direct path

$$S_R(t) = A_d \cdot u(t-\tau) - A_d\, u(t-\tau-T_p)$$
$$+ A_R\, u(t-\tau-\delta\tau) - A_r\, u(t-\tau-\delta\tau-T_p)$$

R is reflected path

Linearity property – break up the terms

Delay property – work with delays

$u(t) \longrightarrow \frac{1}{s}$ unit step

$u(t-\tau) \longrightarrow e^{-\tau s} \times \frac{1}{s}$ delayed unit step

$$F(s) = A_d e^{-\tau s} \times \frac{1}{s} - A_d e^{-(\tau + T_p)s} \times \frac{1}{s}$$
$$+ A_R e^{-(\tau + \delta\tau)s} \times \frac{1}{s} - A_R e^{-(\tau + \delta\tau + T_p)s}$$

$$= \frac{1}{s} e^{-\tau s} \left[A_d - A_d e^{-T_p \cdot s} + A_r e^{-\delta\tau s} - A_R e^{-(\delta\tau + T_p)s} \right]$$

- So we transmitted a pulse and received two pulses
- we computed the Laplace transform of the received signal

In the next few lectures we'll see how useful this is!